# Standard-Model – Theory Part

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**Abstract**

We present a "reading help" to the book "Modern Particle Physics" by Mark Thomson (Cambridge University Press, 2013). These notes cover the theory part of the lecture series "Standard Model" for physics master students at the University of Wuppertal.

# Contents

# 1 Introduction

## 1.1 Quarks and leptons in three generations

The Standard Model (of particle physics) is built as a quantum field theory (QFT). It knows about matter fields (which have spin $\frac{1}{2}$), force fields (which have spin 1), and one field which stays a little apart (the Higgs field with spin 0). In fact, in a QFT the distinction between matter fields and force fields is not necessary in the first place.

| | | | | |
|---|---|---|---|---|
| $u$ | $c$ | $t$ | U-quarks (up-type) | $q = +2/3$ |
| $d$ | $s$ | $b$ | D-quarks (down-type) | $q = -1/3$ |
| $\nu_e$ | $\nu_\mu$ | $\nu_\tau$ | uncharged lept./neutrinos | $q = 0$ |
| $e$ | $\mu$ | $\tau$ | charged leptons | $q = -1$ |

The properties of the matter fields are summarized in this table. The first generation comprises the down ($d$) and up ($u$) quarks, along with the electron-neutrino ($\nu_e$) and the electron ($e^-$). They have been arranged such that the more positively charged partner is on top. For unknown reasons there is a second and third generation, where this pattern is repeated.

| | | | | | |
|---|---|---|---|---|---|
| $u$ | $c$ | $t$ | 0.002 | 1.27 | 173 |
| $d$ | $s$ | $b$ | 0.005 | 0.093 | 4.18 |
| $\nu_e$ | $\nu_\mu$ | $\nu_\tau$ | — | — | — |
| $e$ | $\mu$ | $\tau$ | 0.0005 | 0.106 | 1.78 |

Their masses, in units of GeV (see below), are summarized in this table. There is a broad hierarchy in the sense that up-type quarks are heavier than down-type quarks (except for the $u$ and $d$ themselves), and masses grow with the generation number. Due to confinement, quark masses have a technical character and cannot be measured in experiment (we use the $\overline{\mathrm{MS}}$ scheme, with a renormalization scale $\mu = 2\,\mathrm{GeV}$). Upon changing $\mu$ their values receive a common factor; in a strict sense only quark mass ratios are physical. The masses of the charged leptons $e, \mu, \tau$ can be measured. Note that the neutrinos $\nu_e, \nu_\mu, \nu_\tau$ do not have a mass (some linear combinations $\nu_1, \nu_2, \nu_3$ have a mass, but to date only upper bounds are known).

## 1.2 Strong, weak, and electromagnetic force

In GETA we learned that there are three fundamental forces, the strong, weak, and the electromagnetic force. In addition, there is gravity, but in particle physics it is too weak to matter.

The strong force mediates between two particles, if they both carry a color charge. The weak force mediates between two particles, if they both carry a weak charge. The electromagnetic force mediates between two particles, if they both carry an electric charge.

- U-type quarks experience the strong, the weak, and the electromagnetic force
- D-type quarks experience the strong, the weak, and the electromagnetic force
- neutrinos experience only the weak force
- charged leptons experience the weak and the electromagnetic force

These statements can be translated into statements which type of charges are inside a quark and charged/uncharged lepton. From an algebraic viewpoint a $\nu_e$ is basically a $e^-$ with its electric charge taken out (this does not imply that either one is a composite object).

| | strong | electromagnetic | weak | | gravitational |
|---|---|---|---|---|---|
| $g^2/M^2$ | $\leq 1$ | $\sim 1/137.04$ | $10^{-8}$ | | $10^{-37}$ |
| name | gluon | photon | intermediate vector-boson | | graviton |
| symb. | $g$ | $\gamma$ | $W^\pm$ | $Z$ | $G$ |
| mass | 0 | 0 | 80.4 GeV | 91.2 GeV | 0 |
| spin | 1 | 1 | 1 | 1 | 2 |

The properties of the forces are summarized in this table. The first line indicates the force (in natural units, see below) experienced by two objects carrying a unit charge each, separated by $1\,\mathrm{fm} = 10^{-15}\,\mathrm{m}$. For the strong and electromagnetic force this is equivalent to a statement about the squared coupling $g_{\mathrm{st}}^2$ and $e^2$, respectively. For the weak force things are more tricky.

The intermediate vector-bosons are massive, and they receive their mass through the Higgs mechanism. The other gauge bosons are exactly massless, and this property is linked to an internal "gauge" symmetry. The $g$, the $\gamma$, and the $W^\pm, Z^0$ have been experimentally verified to have spin 1. The spin of the graviton is conjectured to be 2, based on the properties of classical gravitational waves (which are known to exist, and that is quite something).

- gluons couple to color charges, they carry a charge/anti-charge, the gauge group is $SU(3)_C$
- intermediate vector bosons behave similarly with respect to weak charges, the group is $SU(2)_L$
- photons couple to electric charges, they do not carry an electric charge, the group is $U(1)_Q$

The self-coupling property of gluons and intermediate vector-bosons makes these theories more involved, they are based on non-abelian groups. We shall start with Feynman rules for a theory with photons only, known as QED. Later, we will extend them to QCD (quantum chromo-dynamics) and finally to QFD (quantum flavor-dynamics).

## 1.3 Feynman diagrams

A Feynman diagram represents an amplitude $\in \mathbb{C}$ for a process (with well-defined in and out states) to happen via a specified set of internal/intermediate states. Following the principles of quantum mechanics, this amplitude interferes with the amplitudes of processes with the same in and out states but different internal (unobservable) states. In other words, this amplitude is added to the amplitudes of other Feynman diagrams, to give the amplitude of the process (to a given order in an expansion to be discussed below), and the result is absolute-squared to give (up to a constant) the probability for this process to happen.

Consider a slight generalization of the two-slit experiment with two intermediate layers, the first of which has 2 slits, the second of which has 3 slits. To the left of everything there is a given initial point $I$, and to the right of everything there is a given final point $F$. Following the logics of quantum mechanics, there are in total 6 different paths leading from $I$ to $F$. Each one comes with its own amplitude $A_i$, and we are supposed to add the six amplitudes to get the overall amplitude $A = A_1 + ... + A_6$. In the end we take the absolute square to get the probability $P = |A|^2 = |A_1 + ... + A_6|^2$ for the process or path $I \to F$ to happen.

In GETA we learned that a Feynman diagram corresponds to a path in this analogy. On several occasions we drew several Feynman diagrams with a given initial state $I$ and final state $F$, and stated that the numbers $\in \mathbb{C}$ need to be added and absolute squared. So, in this analogy each Feynman diagram corresponds to a path.

The new part in the Standard Model course is that we will discuss how the number that each Feynman diagram represents comes about. We will learn that each line and each vertex in the $i$-th Feynman diagram represents a complex number $a_{ij}$, and the amplitude $A_i$ is the product $\prod_j a_{ij}$, up to an overall factor.

If you reconsider the two-layer multi-slit experiment, you notice that each one of the six amplitudes $A_i$ is made up from three sub-amplitudes, $A_i = a_{i1}a_{i2}a_{i3}$. Obviously, $a_{i1}$ is the sub-amplitude from the source $I$ to the first layer, $a_{i2}$ the one between the two layers, and $a_{i3}$ from the second layer to $F$. This product decomposition of $A_i$ corresponds to the Feynman rules that generate the amplitude of a given Feynman diagram from a set of factors that each one of its lines and vertices stands for.

## 1.4 Standard Model vertices

You should be familiar with the vertices of the Standard Model; in particular whether or not a flavor change takes place (difference between $W$ and $Z$). The content of Fig.1.4 in the book is key to everything that follows in this course. The coupling at the electromagnetic vertex is usually called $e$ or $g_{\text{em}}$; one might call it $g_Q$, to refer to the group $U(1)_Q$. The coupling at the strong vertex is called $g_{\text{S}}$ or $g_{\text{st}}$; one might call it $g_C$, to refer to the group $SU(3)_C$. The couplings at the weak vertices are called $g_W$ and $g_Z$, respectively; they are modified versions of the weak coupling $g$ which one might call $g_L$ to refer to the gauge group $SU(2)_L$.

## 1.5 Mesons and Baryons

Mesons have a net quark content $q\bar{q}$, where either $q$ is a quark, but they need not be identical, e.g. $D_s^+ = (c\bar{s})$. These quarks are bound together by the strong force which also creates lots of "virtual" $q\bar{q}$ pairs (which means that this time $q$ and $\bar{q}$ are the same flavor). We will learn in this course how such virtual quark loops contribute to the Feynman diagram (and hence the amplitude) of a given process.

Baryons have a net quark content $qqq$, where all three $q$ may be different. Apart from its flavor content, each $q$-field has a (suppressed) color index, and the net color must transform as a singlet. The net quark content does not fully specify the particle; for instance $uud$ could be a proton or another particle with the same quark content. What matters is whether the flavors are in a symmetric, antisymmetric, or mixed representation (see below). Again, there are plenty of virtual quark-antiquark pairs, but they do not modify such group related particle properties, since they transform as color and flavor singlets.

A meson can be its own antiparticle, a baryon cannot. The freedom to put the quarks into various representations creates a wealth of possibilities for the resulting mesons and baryons. You should be familiar with the main contents of Tables C1/C2 in App. C of the book.

## 1.6 Summary

From a theory viewpoint, the most important part of Chap. 1 in the book is section 1.1 and the problems. Read everything carefully, and try to solve the problems.

# 2 Prerequisites

## 2.1 Natural units

In particle physics (especially particle theory), one uses "natural units". One sets $\hbar = c = 1$, and what this means is that formulas are written "modulo factors of $\hbar$ and $c$". In other words, all factors of $\hbar$ and $c$ are omitted, since they can be reconstructed in a unique manner.

With this simplification, energy, momentum and mass are all measured in units of $\text{MeV} = 10^6\,\text{eV}$ or $\text{GeV} = 10^9\,\text{eV}$. And times and distances are measured in inverse MeV or inverse GeV. To convert to SI units, one uses $\hbar = 1.055\,10^{-34}\,\text{Js}$ and $c = 2.998\,10^8\,\text{m/s}$. With $\text{J} = \text{W s} = \text{kg}\,\text{m}^2\,\text{s}^{-2}$ and $1\,\text{MeV} = 1.602\,10^{-13}\text{J}$ it follows that $\hbar c = 197.327\,\text{MeV fm}$, and this is something to remember. This implies $(1\,\text{fm})^{-1} \simeq 197\,\text{MeV}$ or $(1\,\text{GeV})^{-1} \simeq 0.197\,\text{fm}$.

| quantity | [kg, m, s] | [$\hbar$, $c$, GeV] | natural units |
|---|---|---|---|
| action | kg m$^2$ s$^{-1}$ | $\hbar$ | 1 |
| energy | kg m$^2$ s$^{-2}$ | GeV | GeV |
| momentum | kg m s$^{-1}$ | GeV/$c$ | GeV |
| mass | kg | GeV/$c^2$ | GeV |
| time | s | (GeV/$\hbar$)$^{-1}$ | GeV$^{-1}$ |
| length | m | (GeV/[$\hbar c$])$^{-1}$ | GeV$^{-1}$ |
| area | m$^2$ | (GeV/[$\hbar c$])$^{-2}$ | GeV$^{-2}$ |
| speed | m s$^{-1}$ | $c$ | 1 |

One unit is hardly used outside of particle physics: $1\,\text{b} = 1\,\text{barn} = (10^{-14}\,\text{m})^2 = 100\,\text{fm}^2$.

## 2.2 Special relativity

You should be familiar with the concept of a "reference frame" which is also called "inertial system". The definition says that a particle which is not subject to a force will move uniformly in this coordinate system. In the old (unprimed) frame a space-time point has the coordinates $r \equiv (ct, x, y, z)^{\text{t}}$. The same space-time point is given in the primed frame by $r' \equiv (ct', x', y', z')^{\text{t}}$. The new frame $S'$ moves in $S$ with speed $v$, e.g. along the $z$-axis, with $v < c$. Hence $\beta \equiv v/c < 1$ and $\gamma \equiv (1-\beta^2)^{-1/2} > 1$. A four-vector is *not* a collection of four scalars, but a collection of four objects which transform in a specific way as we change the frame.

A *contravariant* four-vector transforms like the four coordinates $(r^\mu) \equiv (r^\bullet) \equiv ([c]t, x, y, z)^{\text{t}}$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & & & -\beta\gamma \\ & 1 & & \\ & & 1 & \\ -\beta\gamma & & & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \iff r'^\mu = \Lambda^\mu{}_\nu r^\nu \iff (r'^\bullet) = (\Lambda^\bullet{}_\bullet)(r^\bullet) \tag{2.1}$$

and the inverse transform is easily found to be [so we conclude $(\Lambda_\bullet{}^\bullet) = (\Lambda^\bullet{}_\bullet)^{-1}$ and vice versa]

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & & & \beta\gamma \\ & 1 & & \\ & & 1 & \\ \beta\gamma & & & \gamma \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} \iff r^\rho = \Lambda_\sigma{}^\rho r'^\sigma \iff (r^\bullet) = (\Lambda_\bullet{}^\bullet)(r'^\bullet) \tag{2.2}$$

where $\gamma^2 - \beta^2\gamma^2 = 1$ is used.

A *covariant* four-vector transforms like the four coordinates $(r_\mu) \equiv (r_\bullet) \equiv ([c]t, -x, -y, -z)^{\mathrm{t}}$

$$\begin{pmatrix} t' \\ -x' \\ -y' \\ -z' \end{pmatrix} = \begin{pmatrix} \gamma & & & \beta\gamma \\ & 1 & & \\ & & 1 & \\ \beta\gamma & & & \gamma \end{pmatrix} \begin{pmatrix} t \\ -x \\ -y \\ -z \end{pmatrix} \iff r'_\mu = \Lambda_\mu{}^\nu r_\nu \iff (r'_\bullet) = (\Lambda_\bullet{}^\bullet)(r_\bullet) \tag{2.3}$$

and the inverse transform for covariant components is

$$\begin{pmatrix} t \\ -x \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} \gamma & & & -\beta\gamma \\ & 1 & & \\ & & 1 & \\ -\beta\gamma & & & \gamma \end{pmatrix} \begin{pmatrix} t' \\ -x' \\ -y' \\ -z' \end{pmatrix} \iff r_\rho = \Lambda^\sigma{}_\rho r'_\sigma \iff (r_\bullet) = (\Lambda^\bullet{}_\bullet)(r'_\bullet) \tag{2.4}$$

where again $\gamma^2 - \beta^2\gamma^2 = 1$ is used.

Note that $\det(\Lambda) = \det(\Lambda^\bullet{}_\bullet) = 1$. Also $\det(\Lambda^{-1}) = \det(\Lambda_\bullet{}^\bullet) = 1$. In fact, mathematicians say $\Lambda \in \mathrm{SO}(1,3)$, hence $\Lambda$ is a hyperbolic rotation in the four-dimensional space with "pseudometric" $\mathrm{diag}(1,-1,-1,-1)$. Specifically, there is an angle $\xi$, called *rapidity*, such that

$$\Lambda \equiv (\Lambda^\bullet{}_\bullet) = \begin{pmatrix} \cosh(\xi) & & & -\sinh(\xi) \\ & 1 & & \\ & & 1 & \\ -\sinh(\xi) & & & \cosh(\xi) \end{pmatrix}, \quad \Lambda^{-1} \equiv (\Lambda_\bullet{}^\bullet) = \begin{pmatrix} \cosh(\xi) & & & \sinh(\xi) \\ & 1 & & \\ & & 1 & \\ \sinh(\xi) & & & \cosh(\xi) \end{pmatrix} \tag{2.5}$$

and the relation is $\gamma = \cosh(\xi)$, $\beta = \sinh(\xi)/\gamma = \tanh(\xi)$ or $\xi \equiv \mathrm{arctanh}(\beta) = \frac{1}{2}\log(\frac{1+\beta}{1-\beta})$. The standard "velocity superposition" $\beta = \beta_2 \circ \beta_1$ is really the (much simpler) addition $\xi = \xi_2 + \xi_1$.

A key statement is that the eigentime interval is a Lorentz scalar, which is to say that

$$\Delta s^2 \equiv (\Delta s)^2 \equiv \left\{ \begin{array}{l} c^2\Delta t^2 \ - \Delta x^2 \ - \Delta y^2 \ - \Delta z^2 \\ c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2 \end{array} \right. \tag{2.6}$$

is the same in every inertial frame (IF). This $\Delta s^2$ can be $> 0$, $= 0$, or $< 0$, and we say the two events are separated *timelike*, *lightlike*, or *spacelike*, respectively. In the first case there is an IF where they happen at the same spatial point (so that $\Delta z' = 0$). In the last case there is an IF where they happen at the same time (so that $\Delta t' = 0$). Furthermore,

$$\frac{ds^2}{c^2} = \left\{ \begin{array}{ll} dt^2(1 - \frac{dx^2+dy^2+dz^2}{c^2\, dt^2}) = dt^2(1-\beta^2) & \\ d\tau^2(1-0) & [\text{"rest frame"}] \end{array} \right. \tag{2.7}$$

and this means the eigentime interval $d\tau = ds/c$ can be calculated from every frame through

$$\boxed{d\tau \equiv dt\sqrt{1-\beta^2} = dt\sqrt{1-v(t)^2/c^2}} \tag{2.8}$$

and we shall frequently use that $d\tau$ is a Lorentz invariant quantity or "Lorentz scalar".

We use the Einstein summation convention; an index appearing twice – once upstairs and once downstairs – is summed over. We use this technique to define an indefinite product $\langle u, v\rangle \equiv u^0v^0 - \vec{u}\vec{v}$ for two four-vectors $u, v$. The metric is $\eta_{\mu\nu}$ with $\eta_{\bullet\bullet} = (\eta_{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$,

$$ds^2 = \eta_{\mu\nu} ds^\mu ds^\nu = (ds^0)^2 - (ds^1)^2 - (ds^2)^2 - (ds^3)^2\ . \tag{2.9}$$

The inverse metric is $\eta^{\mu\nu}$, with $\eta^{\bullet\bullet} = (\eta^{\mu\nu}) = [\mathrm{diag}(1,-1,-1,-1)]^{-1} = \mathrm{diag}(1,-1,-1,-1)$, so

$$ds^2 = \eta^{\mu\nu} ds_\mu ds_\nu = (ds^0)^2 - (ds^1)^2 - (ds^2)^2 - (ds^3)^2 \ . \tag{2.10}$$

Now we use this technique to write $ds^2$ both in the unprimed and primed frame

$$ds^2 = \begin{cases} \eta_{\mu\nu} dr^\mu dr^\nu = \eta_{\rho\sigma} dr^\rho dr^\sigma \\ \eta_{\mu\nu} dr'^\mu dr'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\rho dr^\rho \Lambda^\nu{}_\sigma dr^\sigma \end{cases} \tag{2.11}$$

and from comparing the two right-hand sides we conclude $\eta_{\rho\sigma} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu}$. The pseudo-metric $\eta_{\bullet\bullet} = (\eta_{\mu\nu})$ is used to "pull down" indices, and $\eta^{\bullet\bullet} = (\eta^{\mu\nu})$ is used to "pull up" indices. Let $t^\mu{}_\nu$ be a tensor-field with a contravariant (first) and a covariant (second) index. Then $t_\rho{}^\sigma = \eta_{\rho\mu}\eta^{\sigma\nu} t^\mu{}_\nu$. Finally, there is the Kronecker symbol $\delta_\alpha^\beta$ which is 1 for $\alpha = \beta$ and 0 otherwise. Note this object is not a tensor, and this is why we let it have its two indices atop/below each other; for a tensor we must maintain the information on the order of its indices.

In short a tensor is an object which transforms like a dyadic product of contravariant or covariant vectors. The field-strength tensor in classical electrodynamics comes in four varieties, $F^{\bullet\bullet}$, $F_\bullet{}^\bullet$, $F^\bullet{}_\bullet$, $F_{\bullet\bullet}$. They relate to each other via $\eta$-operations, and they transform as

$$F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma} \ , \ F'_\mu{}^\nu = \Lambda_\mu{}^\rho \Lambda^\nu{}_\sigma F_\rho{}^\sigma \ , \ F'^\mu{}_\nu = \Lambda^\mu{}_\rho \Lambda_\nu{}^\sigma F^\rho{}_\sigma \ , \ F'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma} \ . \tag{2.12}$$

A tensor of rank 4 with 3 contravariant and 1 covariant indices would transform as

$$T'^{\mu\nu\rho}{}_\sigma = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda_\sigma{}^\delta T^{\alpha\beta\gamma}{}_\delta \tag{2.13}$$

and a tensor of rank 0 requires no $\Lambda$-factor at all (so it is a Lorentz scalar like $d\tau$ or $d\tau^2$).

Last but not least four-derivatives have specific transformation properties. For a boost in $z$-direction with $\Lambda^\bullet{}_\bullet$ as above the derivatives in the primed frame are given by

$$\partial_{z'} = \Big(\frac{\partial z}{\partial z'}\Big)\partial_z + \Big(\frac{\partial t}{\partial z'}\Big)\partial_t \ , \quad \partial_{t'} = \Big(\frac{\partial z}{\partial t'}\Big)\partial_z + \Big(\frac{\partial t}{\partial t'}\Big)\partial_t \ , \tag{2.14}$$

and thanks to the linearity of the Lorentz boots the parentheses can be read off from the inverse transformation (2.2), so $\partial_{z'} = \gamma\partial_z + \beta\gamma\partial_t$ and $\partial_{t'} = \beta\gamma\partial_z + \gamma\partial_t$. We summarize this in the form

$$\begin{pmatrix} \partial_{t'} \\ \partial_{x'} \\ \partial_{y'} \\ \partial_{z'} \end{pmatrix} = \begin{pmatrix} \gamma & & & \beta\gamma \\ & 1 & & \\ & & 1 & \\ \beta\gamma & & & \gamma \end{pmatrix} \begin{pmatrix} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \tag{2.15}$$

and we emphasize that this is different from (2.1). Hence we can define a derivative operator

$$\partial_\bullet \equiv (\partial_\mu) \equiv \Big(\frac{\partial}{\partial x^\mu}\Big) = \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \tag{2.16}$$

which transforms as a *covariant* vector, and another derivative operator

$$\partial^\bullet \equiv (\partial^\mu) \equiv \Big(\frac{\partial}{\partial x_\mu}\Big) = \begin{pmatrix} \frac{\partial}{\partial t} \\ -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \partial_t \\ -\partial_x \\ -\partial_y \\ -\partial_z \end{pmatrix} \tag{2.17}$$

which transforms as a *contravariant* vector. At first sight it might be perplexing that the derivatives w.r.t. the contravariant components transform as a covariant vector, and the derivatives w.r.t. the covariant components as a contravariant vector. Next we can use these objects to define a second-order derivative operator (known as d'Alembert or wave-equation operator)

$$\Box \equiv \partial_\mu \partial^\mu = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \tag{2.18}$$

which indeed transforms as a Lorentz scalar, i.e. it looks exactly the same in every inertial frame, as its construction via contraction of a covariant and a contravariant index would suggest.

Given that $x^\mu$ is a four-vector, and only $d\tau$ (but not $dt$) is a Lorentz scalar, it is clear that a valid four-velocity (which again transforms as a contravariant four-vector) must be defined as

$$u^\mu \equiv \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \gamma(v)\begin{pmatrix} c \\ \vec{v} \end{pmatrix} = \gamma(\beta)\begin{pmatrix} 1 \\ \vec{\beta} \end{pmatrix} \tag{2.19}$$

where $dt = \gamma d\tau$, see (2.8), has been used. With $u^2 \equiv u_\mu u^\mu = \gamma^2(1-\beta^2) = 1$ it follows that any valid four-velocity has *unit length* in the Einstein pseudometric.

Given $u^\mu$, one defines the four-momentum as $p^\mu = mu^\mu$, with $m$ the *rest mass* of the particle, and it follows that $p^2 = m^2$, or $p^2 = (mc)^2$ with $c$ restored. Note that the square on $p$ is a pseudometric square. With $p^\bullet = m\gamma(c,\vec{v})^\mathrm{t}$ and $p_\bullet = m\gamma(c,-\vec{v})^\mathrm{t}$ we have $p^2 = p^\mu p_\mu = m^2\gamma^2[c^2 - \vec{v}^2] = m^2\gamma^2c^2[1-\beta^2] = (mc)^2$, as promised. The three spatial components of $p^\bullet$ are $m\gamma(v)\vec{v}$, i.e. $\gamma(v)$ times the Newtonian momentum. Its zeroth component

$$E \equiv cp^0 = \gamma(v)mc^2 \tag{2.20}$$

is called the relativistic energy. The fixed-length property of $p^2$ unfolds as

$$\boxed{p^2 = (E/c)^2 - \vec{p}^2 = m^2c^2} \tag{2.21}$$

which is known under the heading "relativistic dispersion relation". Note that it holds true in any frame, with frame-dependent $E$ and $\vec{p}$, but fixed $m$.

A simple application is a decay $0 \to 1+2$ where four-momentum conservation yields the invariant mass square of the original particle,

$$(p^{(1)} + p^{(2)})^\mu (p^{(1)} + p^{(2)})_\mu = (p^{(0)})^\mu (p^{(0)})_\mu = (m^{(0)}c)^2\ . \tag{2.22}$$

Similarly, in a scattering process $1+2 \to 3+4$ one defines the Mandelstam variables

$$\begin{aligned} s &= (p^{(1)} + p^{(2)})^2 = (p^{(3)} + p^{(4)})^2 \\ t &= (p^{(1)} - p^{(3)})^2 = (p^{(2)} - p^{(4)})^2 \\ u &= (p^{(1)} - p^{(4)})^2 = (p^{(2)} - p^{(3)})^2 \end{aligned} \tag{2.23}$$

each of which happens to be a Lorentz scalar. With a little algebra, the sum can be shown to equal the sum of the mass squares, i.e. $s + t + u = (m^{(1)})^2 + (m^{(2)})^2 + (m^{(3)})^2 + (m^{(4)})^2$.

## 2.3 Compatible observables

The key principle of quantum mechanics is the deterministic evolution of the wave function

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(t,\vec{x}) = H\psi(t,\vec{x}) \tag{2.24}$$

known as the "Schrödinger equation". If the Hamiltonian $H$ of the system is $t$-independent, the closed solution reads $\psi(t,\vec{x}) = \exp(-\mathrm{i}Ht/\hbar)\psi(0,\vec{x})$, where $\exp(.)$ is the matrix exponential. In the event $H$ has a complete set of eigenstates, it is advisable to first determine them through

$$H\phi_i(\vec{x}) = E_i\phi_i(\vec{x}) \tag{2.25}$$

which is referred to as "solving the time-independent Schrödinger equation".

In non-relativistic quantum mechanics it is popular to work in position space. If the Hamiltonian decomposes into a kinetic and a potential term, $H = T + V$, the former one is $T = \frac{1}{2m}\vec{P}^2 = \frac{1}{2m}(\frac{\hbar}{\mathrm{i}}\partial_{\vec{x}})^2 = -\frac{\hbar^2}{2m}\triangle$, and the potential is usually given in $x$-space, hence

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(t,\vec{x}) = -\frac{\hbar^2}{2m}\triangle\psi(t,\vec{x}) + V(\vec{x})\psi(t,\vec{x})\;. \tag{2.26}$$

Alternatively, one might consider things in momentum space. In this case the kinetic term is just a left-multiplication with $\frac{1}{2m}\vec{p}^2$, but now the potential term is complicated, since it needs to be Fourier transformed, and every factor $\vec{x}$ must be emulated through $\mathrm{i}\hbar\partial_{\vec{p}}$. In short

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{\psi}(t,\vec{p}) = \frac{1}{2m}\vec{p}^2\hat{\psi}(t,\vec{p}) + V(\mathrm{i}\hbar\partial_{\vec{p}})\hat{\psi}(t,\vec{p}) \tag{2.27}$$

is the Schrödinger equation in momentum space. Note that the hat denotes Fourier transform w.r.t. the (one or three) spatial coordinates; the time $t$ is a mere parameter.

Given the indefinite scalar product $\langle u,v\rangle = u_\mu v^\mu = u^0v^0 - \vec{u}\cdot\vec{v}$ in the previous subsection, it makes sense to adopt different conventions for the Fourier transform in the $\vec{x} \leftrightarrow \vec{k}$ pair and the $t \leftrightarrow \omega$ pair. We are used to $\psi(\vec{x}) = \mathrm{const}\int\hat{\psi}(\vec{k})e^{\mathrm{i}\vec{k}\cdot\vec{x}}d^3k$, and we want to keep this convention. Therefore we shall choose the opposite sign in the exponential for the other pair, that is $\psi(t) = \mathrm{const}\int\hat{\psi}(\omega)e^{-\mathrm{i}\omega t}d\omega$. Sometimes people use a tilde rather than a hat for the time-frequency transformation, but we shall not do so, since we shall combine both transformations in a minute. What is not a convention is that the reverse transformation comes with the opposite sign. Hence, once we have adopted these conventions for $\psi(\vec{x})$ and $\psi(t)$, we must use the opposite signs in $\hat{\psi}(\vec{k}) = \mathrm{const}\int\psi(\vec{x})e^{-\mathrm{i}\vec{k}\cdot\vec{x}}d^3x$ and $\hat{\psi}(\omega) = \mathrm{const}\int\psi(t)e^{\mathrm{i}\omega t}dt$. Having fully specified our conventions, we may combine the two transformations as

$$\begin{aligned}
\psi(t,\vec{x}) &= \mathrm{const}\int\hat{\psi}(\omega,\vec{k})\,e^{\mathrm{i}[\vec{k}\cdot\vec{x}-\omega t]}\,d\omega\,d^3k = \mathrm{const}\int\hat{\psi}(\omega,\vec{k})\,e^{-\mathrm{i}\langle k,x\rangle}\,d\omega\,d^3k & (2.28)\\
\hat{\psi}(\omega,\vec{k}) &= \mathrm{const}\int\psi(t,\vec{x})\,e^{-\mathrm{i}[\vec{k}\cdot\vec{x}-\omega t]}\,dt\,d^3x = \mathrm{const}\int\psi(t,\vec{x})\;e^{+\mathrm{i}\langle k,x\rangle}\,dt\,d^3x & (2.29)
\end{aligned}$$

and the factor $e^{\pm\mathrm{i}[\vec{k}\cdot\vec{x}-\omega t]}$ can be rewritten as $e^{\pm\mathrm{i}[\vec{p}\cdot\vec{x}-Et]/\hbar}$, since $\vec{p} = \hbar\vec{k}$ ("DeBroglie relation") and $E = \hbar\omega = h\nu$ ("Einstein relation"), with $|\vec{k}| = 2\pi/\lambda$ and $\lambda\nu = c$ for photons.

An observable is represented by a hermitean operator, say $A = A^\dagger$. The measurement in a state $\psi$ amounts to $\int\psi^* A\psi$, where the star denotes complex conjugation. The integral is over

$x$ or $p$, depending on whether $\psi$ is given in ordinary or momentum space. A universal notation is given by the bra-ket notation of Dirac. In case of two different states it reads

$$\langle\psi|A|\phi\rangle = \int \psi(x)^* A\phi(x)\, dx = \int \psi(p)^* A\phi(p)\, dp \tag{2.30}$$

where the Fourier hat is omitted to make it look more symmetric. The hermiticity (self-adjointness) of $A$ results in $\langle A\psi|\phi\rangle = \langle\psi|A\phi\rangle = \int [A\psi(x)]^*\phi(x)\, dx = \int \psi(x)^* A\phi(x)\, dx$.

If $\phi$ is an eigenstate of $A$, i.e. $A\phi = a\phi$, the measured value is the eigenvalue $a$, and the operator is said to be "sharp" on this specific state. If $A$ has several eigenvalues, the respective eigenstates are orthogonal (or can be made orthogonal if the eigenvalue is degenerate). For two observables $A$ and $B$ the question emerges whether they can be simultaneously "sharp". The key statement is (with $[A, B] \equiv AB - BA$ the commutator of $A$ and $B$)

$$[A, B] = 0 \;\longleftrightarrow\; A, B\,\text{compatible} \;\longleftrightarrow\; A, B\,\text{have simultaneous eigenvalues}\, a, b \tag{2.31}$$

and the proof is found in the book. The bottom line can be formulated as follows: *quantum states are labeled by the quantum numbers of a maximal set of compatible operators.* A typical example in atomic physics is based on $[L^2, L_z] = 0$; so states are labeled as $|\ell, m\rangle$.

## 2.4 Angular momentum

The angular momentum operator has three components, each of which is a bilinear combination of the respective components of the position operator $\vec{R}$ and momentum operator $\vec{P}$

$$\vec{L} = \vec{R} \wedge \vec{P} = \begin{pmatrix} YP_z - ZP_y \\ ZP_x - XP_z \\ XP_y - YP_x \end{pmatrix} \tag{2.32}$$

where both $\vec{R}$ and $\vec{P}$ are generic. Only after we decide to work in position/momentum space one of them acts multiplicatively, while the other one acts through a derivative

$$\begin{aligned} &\text{pos. space:} \quad \vec{P} = \frac{\hbar}{\mathrm{i}}\nabla_{\vec{r}} \doteq -\mathrm{i}\nabla_{\vec{r}} \quad \longrightarrow \quad L_x = yP_z - zP_y \doteq -\mathrm{i}y\partial_z + \mathrm{i}z\partial_y \\ &\text{mom. space:} \quad \vec{R} = \mathrm{i}\hbar\nabla_{\vec{p}} \doteq +\mathrm{i}\nabla_{\vec{p}} \quad \longrightarrow \quad L_x = Yp_z - Zp_y \doteq +\mathrm{i}p_z\partial_{p_y} - \mathrm{i}p_y\partial_{p_z} \end{aligned} \tag{2.33}$$

and with either choice one verifies the basic commutation relations (3 are non-zero, 6 are zero)

$$[X, P_x] = [Y, P_y] = [Z, P_z] \doteq \mathrm{i}\;, \quad [X, P_y] = ... = 0\;. \tag{2.34}$$

Based on these relations it is straightforward to establish the key relations

$$[L_x, L_y] \doteq \mathrm{i}L_z\;, \quad [L_y, L_z] \doteq \mathrm{i}L_x\;, \quad [L_z, L_x] \doteq \mathrm{i}L_y \tag{2.35}$$

with obvious cyclic behavior. The commutator relations define the *algebra*; this is relevant to particle physics, since the same algebra will show up in the context of flavor quantum numbers.

An important operator is the angular momentum squared operator, defined as

$$L^2 \equiv L_x^2 + L_y^2 + L_z^2 \tag{2.36}$$

and it is easy to verify that it commutes with each component of $\vec{L}$, that is

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0 \ . \tag{2.37}$$

However, since these three operators do not commute with each other, a state is defined/labeled by the quantum numbers of $L^2$ and *any one* of $L_x$, $L_y$ or $L_z$. The standard choice is to use $[L^2, L_z] = 0$, and this results in the quantum numbers $\ell$ and $m \equiv m_z$.

Having adopted the convention of using $L_z$ for the quantum number $m$ (so that $m$ defines the "sharp" value of the angular momentum projection $\vec{L} \cdot \vec{e}_z$), one defines the "ladder operators"

$$\begin{aligned} L_+ &= L_x + \mathrm{i}L_y \qquad (\text{"raising operator"}) & (2.38) \\ L_- &= L_x - \mathrm{i}L_y \qquad (\text{"lowering operator"}) & (2.39) \end{aligned}$$

and it is easy to verify that either one commutes with the angular momentum squared operator

$$[L^2, L_\pm] = ... = 0 \ . \tag{2.40}$$

It is even easier to work out the commutation relation with $L_z$

$$[L_z, L_\pm] = [L_z, L_x] \pm \mathrm{i}[L_z, L_y] = \mathrm{i}L_y \pm \mathrm{i}(-\mathrm{i}L_x) = \mathrm{i}L_y \pm L_x = \pm L_\pm \tag{2.41}$$

and slightly more tedious to work out the decomposition of $L^2$

$$L^2 = ... = L_- L_+ + L_z + L_z^2 \ . \tag{2.42}$$

Hence it is clear that one should base the labeling of the states on the choice $L^2, L_z$ to define the quantum numbers, and observe the effect of $L_\pm$ on $|\ell, m\rangle$ with $-\ell \leq m \leq \ell$, in short

$$\begin{aligned} L^2|\ell, m\rangle &= \ell(\ell+1)|\ell, m\rangle & (2.43) \\ L_z|\ell, m\rangle &= m|\ell, m\rangle & (2.44) \\ L_+|\ell, m\rangle &= ... = \sqrt{\ell(\ell+1) - m(m+1)}|\ell, m+1\rangle \quad \text{for} \quad m < +\ell \quad (\text{otherwise}\, 0) & (2.45) \\ L_-|\ell, m\rangle &= ... = \sqrt{\ell(\ell+1) - m(m-1)}|\ell, m-1\rangle \quad \text{for} \quad m > -\ell \quad (\text{otherwise}\, 0) \ . & (2.46) \end{aligned}$$

Finally we recall that spin $\frac{1}{2}$ is represented in quantum mechanics through the Pauli matrices

$$\sigma_1 \equiv \sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 \equiv \sigma_y \equiv \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \ , \quad \sigma_3 \equiv \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{2.47}$$

with $S_i = \frac{\hbar}{2}\sigma_i$. The commutator relations $[S_i, S_j] = \epsilon_{ijk}\mathrm{i}\hbar S_k$ (cyclic behavior) follow from

$$\left[\frac{\sigma_x}{2}, \frac{\sigma_y}{2}\right] = \mathrm{i}\frac{\sigma_z}{2} \ , \qquad \left[\frac{\sigma_y}{2}, \frac{\sigma_z}{2}\right] = \mathrm{i}\frac{\sigma_x}{2} \ , \qquad \left[\frac{\sigma_z}{2}, \frac{\sigma_x}{2}\right] = \mathrm{i}\frac{\sigma_y}{2} \ . \tag{2.48}$$

## 2.5 Perturbative expansion

Sometimes the Hamiltonian can be separated into a "big" (and time-independent) contribution $H_0$ which is easy to deal with, and a "small" (and possibly time dependent) contribution $H_{\text{int}}$ which makes the full problem hard to deal with, say

$$H = H_0 + H_{\text{int}} \quad \text{with} \quad H_0 = H_0(\vec{x}) \quad \text{and} \quad H_{\text{int}} = H_{\text{int}}([t,]\vec{x}) \tag{2.49}$$

where $|||H_0||| \gg |||H_{\text{int}}|||$ in any reasonable operator norm $|||...|||$. This $H_{\text{int}}([t,]\vec{x})$ is called the "interaction Hamiltonian". The full Schrödinger equation is thus

$$\mathrm{i}\frac{d}{dt}\psi(t,\vec{x}) = \Big[H_0(\vec{x}) + H_{\text{int}}(t,\vec{x})\Big]\psi(t,\vec{x}) \tag{2.50}$$

and the trick is to use the eigensystem $\phi_i(\vec{x})$ of the *unperturbed* part $H_0(\vec{x})$.

The transition matrix element from initial state $|i\rangle$ to final state $|f\rangle$ is defined as

$$T_{fi} \equiv \langle f|H(t,\vec{x})|i\rangle = \int \phi_f^*(\vec{x})H(t,\vec{x})\phi_i(\vec{x})\,d^3x = \langle f|H|i\rangle \tag{2.51}$$

and it inherits the dimension of energy from $H$, since $\langle f|i\rangle \in \mathbb{C}$. A crude approximation is

$$T_{fi} \simeq \langle f|H_{\text{int}}(t,\vec{x})|i\rangle = \int \phi_f^*(\vec{x})H_{\text{int}}(t,\vec{x})\phi_i(\vec{x})\,d^3x = \langle f|H_{\text{int}}|i\rangle \tag{2.52}$$

which is the leading order in perturbation theory. A better approximation is

$$T_{fi} \simeq \langle f|H_{\text{int}}|i\rangle - \sum_{j\neq i}\frac{\langle f|H_{\text{int}}|j\rangle\langle j|H_{\text{int}}|i\rangle}{E_j - E_i} \tag{2.53}$$

which is the second order in perturbation theory. The sum is over all intermediate states $j$ which are different from the initial state $i$. It takes little imagination to see that this perturbative series can be systematically improved. The next term would involve two sets of intermediate states, enumerated as $k$ and $j$, along with the energy differences $E_k - E_j$ and $E_j - E_i$.

No matter at which order in perturbation theory one has determined the matrix element $T_{fi}$, its modulus square is proportional to the transition rate $\Gamma_{fi}$. Fermi's golden rule says

$$\boxed{\Gamma_{fi} = 2\pi|T_{fi}|^2\rho(E_i)} \tag{2.54}$$

where $\rho(E_i) = \frac{dn}{dE_f}|_{E_i}$ is the density of states in the out-Fockspace (see QFT course). These "phase space" considerations are presented in Chap. 3 of the book. We skip it in the interest of time, but it is important to any experiment in particle physics.

## 2.6 Mathematics

- You should be familiar with linear algebra, including elementary spectral representation.
- You should be familiar with full calculus, including complex functions and contour integrals.
- You should be familiar with distributions, see e.g. App. A of the book.
- You should be familiar with the (complex) Fourier transform.
- You should be familiar with the concept of Green functions, e.g. from TP2.

## 2.7 Summary

From a theory viewpoint, all three sections of Chap. 2 in the book are relevant. Read everything carefully, and try to solve the problems.

# 3 Relativistic field equations

## 3.1 Klein-Gordon equation

The Klein-Gordon equation is the relativistic successor of the Schrödinger equation. The latter equation is based on the dispersion relation $E = \vec{p}^2/(2m)$, so the former one must be based on $E^2 = \vec{p}^2 + m^2$. Thinking of $\psi(x) \propto e^{\mathrm{i}[\vec{p}\vec{x}-Et]}$ we have $\partial_t \sim -\mathrm{i}E$ and $\vec{\nabla} \equiv \partial_{\vec{x}} \sim \mathrm{i}\vec{p}$, hence

$$\boxed{\big(\Box + m^2\big)\psi(t,\vec{x}) = 0} \qquad \text{with} \qquad \Box \equiv \partial_\mu \partial^\mu \equiv \frac{d^2}{dt^2} - \triangle \equiv \partial_t^2 - \sum_{k=1}^{3} \partial_k^2 \tag{3.1}$$

has the required properties ($\Box$, $m^2$ and 0 transform as Lorentz scalars, while $\psi$ is unchanged under a boost, provided the argument $x^\mu$ is replaced by the boosted $x'^\mu = \Lambda^\mu{}_\nu x^\nu$).

There is a problem with the usage of (3.1) as a one-particle wave equation. It implements $E^2 = \vec{p}^2 + m^2$, so mathematically $E = \pm\sqrt{\vec{p}^2 + m^2}$. Physicswise, however, a negative energy makes no sense. In classical mechanics, one can restrict the solution space to the positive square root. In quantum mechanics, one cannot; the solutions must form a complete set of states. These problems manifest themselves in the form of a negative probability density, $\rho(t,\vec{x}) < 0$. Consider $\partial_t^2\psi = \triangle\psi - m^2\psi$ along with $\partial_t^2\psi^* = \triangle\psi^* - m^2\psi^*$. By taking $\psi^*$ times the first minus $\psi$ times the second, one has $\psi^*\partial_t^2\psi - \psi\partial_t^2\psi^* = \psi^*(\vec{\nabla}^2 - m^2)\psi - \psi(\vec{\nabla}^2 - m^2)\psi^* = \psi^*\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\psi^*$. This can be re-written as $\partial_t(\psi^*\partial_t\psi - \psi\partial_t\psi^*) = \vec{\nabla}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)$. Hence, with an extra i,

$$\partial_t\rho(t,\vec{x}) + \vec{\nabla}\vec{j}(t,\vec{x}) = 0 \quad \text{with} \quad \begin{array}{rcl} \rho & \equiv & +\mathrm{i}(\psi^*\partial_t\psi - \psi\partial_t\psi^*) \\ \vec{j} & \equiv & -\mathrm{i}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) \end{array} \tag{3.2}$$

and a plane wave solution has $\rho = 2|N|^2 E \gtrless 0$ and $\vec{j} = 2|N|^2\vec{p}$. The $\rho < 0$ problem has solution within relativistic quantum mechanics. There is a natural solution, though, in the context of a relativistic field theory ("many particle theory", "second quantization"). There, equation (3.1) can, in principle, be used to describe an uncharged pion field or the physical Higgs field (in reality the pion is part of a triplet, and the Higgs is part of a doublet), without interaction.

## 3.2 Dirac equation

Dirac was looking for an equation that would be first-order for both space and time, say

$$\mathrm{i}\partial_t\psi(t,\vec{x}) = \big(-\mathrm{i}\alpha_x\partial_x - \mathrm{i}\alpha_y\partial_y - \mathrm{i}\alpha_z\partial_z + \beta m\big)\psi(t,\vec{x}) \tag{3.3}$$

with unknown objects $\alpha_x, \alpha_y, \alpha_z$ and $\beta$ (later they turn out to be $4\times 4$ matrices). But he requested that $\psi = \psi(x) = \psi(t,\vec{x})$ would *also* satisfy the Klein-Gordon equation (3.1). Hence

$$\begin{aligned} \partial_t^2\psi &= \alpha_x^2\partial_x^2\psi + \alpha_y^2\partial_y^2\psi + \alpha_z^2\partial_z^2\psi - \beta^2 m^2\psi \\ &+ \big\{\alpha_x,\alpha_y\big\}\partial_x\partial_y\psi + \big\{\alpha_y,\alpha_z\big\}\partial_y\partial_z\psi + \big\{\alpha_z,\alpha_x\big\}\partial_z\partial_x\psi \\ &+ \mathrm{i}\big\{\alpha_x,\beta\big\}m\partial_x\psi + \mathrm{i}\big\{\alpha_y,\beta\big\}m\partial_y\psi + \mathrm{i}\big\{\alpha_z,\beta\big\}m\partial_z\psi \\ &\overset{(!)}{=} \partial_x^2\psi + \partial_y^2\psi + \partial_z^2\psi - m^2\psi \end{aligned} \tag{3.4}$$

with $\{A,B\} \equiv AB + BA$ the anti-commutator of $A$ and $B$, and this implies a set of relations

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1\,, \qquad \big\{\alpha_i,\beta\big\} = 0\,, \qquad \big\{\alpha_j,\alpha_k\big\} = 0 \quad (j \neq k) \tag{3.5}$$

among the unknown objects. They imply $\mathrm{tr}(\alpha_i) = 0$, $\mathrm{tr}(\beta) = 0$, and $\alpha, \beta$ can have eigenvalues $\pm 1$ only. Furthermore $\alpha_i^\dagger = \alpha_i$ and $\beta^\dagger = \beta$ follow from the hermiticity of the Dirac Hamiltonian

$$H_{\mathrm{D}} \equiv \vec{\alpha}\vec{P} + \beta m \; . \tag{3.6}$$

The smallest possible representation for $d$ space-time dimensions is $2^{d/2}$ dimensional; for $d = 4$

$$\alpha_i \equiv \sigma_1 \otimes \sigma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \;, \quad \beta \equiv \sigma_3 \otimes I_2 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \tag{3.7}$$

is known as the Dirac-Pauli representation of the $\alpha_i, \beta$ (with $I_2$ the identity in 2 dimensions).

The bottom line is that $\psi(x)$ is not a scalar wavefunction but a spinor field

$$\psi(t, \vec{x}) = \begin{pmatrix} \psi_1(t,\vec{x}) \\ \psi_2(t,\vec{x}) \\ \psi_3(t,\vec{x}) \\ \psi_4(t,\vec{x}) \end{pmatrix} \tag{3.8}$$

with $2^{d/2}$ components (so there is some *internal degree of freedom* to be discussed below).

Later we will encounter other representations, but they relate to (3.7) unitarily, so

$$\tilde{\alpha}_i = U \alpha_i U^\dagger \;, \quad \tilde{\beta} = U \beta U^\dagger \quad \text{with} \quad U \in U(4) \; . \tag{3.9}$$

With $\alpha_i, \beta$ defined in (3.7) we can summarize the Dirac equation in the compact form

$$\boxed{\big(\mathrm{i}\partial_t + \mathrm{i}\alpha_x \partial_x + \mathrm{i}\alpha_y \partial_y + \mathrm{i}\alpha_z \partial_z - \beta m\big)\psi(t,\vec{x}) = 0} \tag{3.10}$$

but it is not yet obvious that this equation is form-invariant under boosts and rotations.

## 3.3 Probability density/current

Repeating the probability density/current consideration of the Klein-Gordon subsection for a field $\psi(t, \vec{x})$ which satisfies the Dirac equation (3.10), one finds

$$\frac{d}{dt}\big(\underbrace{\psi^\dagger \psi}_{\rho(t,\vec{x})}\big) + \vec{\nabla} \cdot \big(\underbrace{\psi^\dagger \vec{\alpha} \psi}_{\vec{j}(t,\vec{x})}\big) = 0 \qquad \text{with} \qquad \vec{\nabla} \cdot \vec{\gamma} \equiv \nabla_x \gamma_x + \ldots + \nabla_z \gamma_z \tag{3.11}$$

and the bottom line is that now $\rho(t, \vec{x}) \geq 0$ is guaranteed, since

$$\rho = \psi^\dagger \psi = \begin{pmatrix} \psi_1^* & \psi_2^* & \psi_3^* & \psi_4^* \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \psi_1^* \psi_1 + \ldots + \psi_4^* \psi_4 = |\psi_1|^2 + \ldots + |\psi_4|^2 \geq 0 \; . \tag{3.12}$$

## 3.4 Spin-1/2 property and Lorentz covariance

In quantum mechanics one has the time evolution property of an expected value

$$\frac{d}{dt}\langle O\rangle = \mathrm{i}\big\langle [H,O]\big\rangle \tag{3.13}$$

where $\langle . \rangle$ refers to a single state $\psi$ or an ensemble of such states (this equation holds both in the Schrödinger picture and in the Heisenberg picture). The $L_i$ are constants of motion, since

$$H_\mathrm{S} \equiv \frac{1}{2m}\vec{P}^2\ , \quad \vec{L} \equiv \vec{R}\wedge\vec{P} \quad \longrightarrow \quad [H_\mathrm{S},\vec{L}] = \vec{0}\ . \tag{3.14}$$

Repeating this calculation for the Dirac Hamiltonian (3.6) one finds

$$[H_\mathrm{D},\vec{L}] = ... = -\mathrm{i}\vec{\alpha}\wedge\vec{P}$$

so any $L_i$ is not conserved. Fortunately, this is not the end of the story; the spin operator

$$\vec{S} \equiv \frac{1}{2}\vec{\Sigma} \equiv \frac{1}{2}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix} \tag{3.15}$$

is not a constant of motion [for a field satisfying the Dirac equation (3.10)] either

$$[H_\mathrm{D},\vec{S}] = ... = +\mathrm{i}\vec{\alpha}\wedge\vec{P}$$

but with opposite sign. Hence we should consider the *total spin* operator

$$\vec{J} \equiv \vec{L}+\vec{S} \quad \Longrightarrow \quad [H_\mathrm{D},\vec{J}] = [H_\mathrm{D},\vec{L}] + [H_\mathrm{D},\vec{S}] = -\mathrm{i}\vec{\alpha}\wedge\vec{P} + \mathrm{i}\vec{\alpha}\wedge\vec{P} = \vec{0} \tag{3.16}$$

to find a constant of motion for a field $\psi(x)$ which satisfies the Dirac equation.

The definition (3.15) means the algebra of $S_1, S_2, S_3$ is identical to the algebra of $\frac{1}{2}\sigma_1$, $\frac{1}{2}\sigma_2$, $\frac{1}{2}\sigma_3$

$$\big[S_1,S_2\big] = \mathrm{i}S_3 \quad \text{(cycl.)}\ . \tag{3.17}$$

Recall that we should label states through the effect of a maximal set of operators. For a Dirac field $\psi(x)$ one chooses $\vec{S}^2$ and $S_z = S_3$, so states are labeled through $|s,m_s\rangle$. Furthermore

$$\vec{S}^2 \equiv S_1^2+S_2^2+S_3^2 = \frac{1}{4}\begin{pmatrix}\sigma_1^2+\sigma_2^2+\sigma_3^2 & 0\\ 0 & \sigma_1^2+\sigma_2^2+\sigma_3^2\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix} = \frac{3}{4}I_4 \tag{3.18}$$

$$\vec{S}^2|s,m_s\rangle = \left\{\begin{array}{cc} s(s+1) & |s,m_s\rangle \\ \frac{3}{4} & |s,m_s\rangle\end{array}\right. \qquad \Longrightarrow \qquad s=\frac{1}{2} \tag{3.19}$$

whereupon the Dirac equation (3.10) is found to describe a spin-1/2 particle.

By acting on the Dirac equation with $\beta$ from the left

$$\mathrm{i}\beta\frac{\partial\psi}{\partial t} + \mathrm{i}\beta\alpha_x\frac{\partial\psi}{\partial x} + ... + \mathrm{i}\beta\alpha_z\frac{\partial\psi}{\partial z} - \beta^2 m\psi = 0 \tag{3.20}$$

$$\gamma^0 \equiv \beta\ , \quad \gamma^1 \equiv \beta\alpha_x \equiv \beta\alpha_1\ , \quad \gamma^2 \equiv \beta\alpha_y \equiv \beta\alpha_2\ , \quad \gamma^3 \equiv \beta\alpha_z \equiv \beta\alpha_3 \tag{3.21}$$

one brings it to a form

$$\big(\mathrm{i}\gamma^0\partial_t + \mathrm{i}\gamma^1\partial_x + ... + \mathrm{i}\gamma^3\partial_z - m\big)\psi(t,\vec{x}) = 0 \tag{3.22}$$

$$\boxed{\big(\mathrm{i}\gamma^\mu\partial_\mu - m\big)\psi(x)} \quad \text{with} \quad x = x^\bullet = (ct,x^1,x^2,x^3)^\mathrm{t} \tag{3.23}$$

which satisfies the requirement of form invariance under boosts/rotations (see book App. B).

## 3.5 Dirac-Clifford algebra

The complex $4 \times 4$ matrices (for $d = 4$ space-time dimensions) satisfy

$$\begin{aligned} (\gamma^1)^2 &= \beta\alpha_x\beta\alpha_x = -\beta^2\alpha_x^2 = -I_4 \\ \gamma^1\gamma^2 &= \beta\alpha_x\beta\alpha_y = -\beta^2\alpha_x\alpha_y = \beta^2\alpha_y\alpha_x = -\beta\alpha_y\beta\alpha_x = -\gamma^2\gamma^1 \end{aligned}$$

and this can be summarized in the algebra

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_4} \, . \tag{3.24}$$

Furthermore, the hermiticity of the $\vec{\alpha}, \beta$ implies the mixed hermiticity/antihermiticity property

$$(\gamma^0)^\dagger = \gamma^0 \quad , \qquad (\gamma^k)^\dagger = -\gamma^k \quad \text{for} \quad k \in \{1, .., 3\} \, . \tag{3.25}$$

Specifically in the Dirac-Pauli representation (3.7) the $\gamma$-matrices take the form

$$\boxed{\gamma^0 \equiv \sigma_3 \otimes I_2 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}} \quad , \qquad \boxed{\gamma^k \equiv \mathrm{i}\sigma_2 \otimes \sigma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}} \tag{3.26}$$

so this is the representation where $\gamma^0 = \mathrm{diag}(1, 1, -1, -1)$ is diagonal. In another representation

$$\tilde{\gamma}^\mu = U\gamma^\mu U^\dagger \quad \text{with} \quad U \in U(4) \tag{3.27}$$

the $\gamma$-matrices would be unitarily related. From this property it follows that the Clifford algebra (3.24) holds for any representation, and a similar statement can be made for the hermiticity/antihermiticity property (3.25).

Upon using the $\gamma$-matrices the probability density and current take the convenient form

$$\left.\begin{aligned} \rho(x) = \psi^\dagger(x)\psi(x) = \psi^\dagger(x)\gamma^0\gamma^0\psi(x) \\ \vec{j}(x) = \psi^\dagger(x)\vec{\alpha}\psi(x) = \psi^\dagger(x)\gamma^0\vec{\gamma}\psi(x) \end{aligned}\right\} \quad \longrightarrow \quad j^\mu(x) \equiv \psi^\dagger(x)\gamma^0\gamma^\mu\psi(x) \tag{3.28}$$

and the continuity equation (for the probability density/current) looks Lorentz covariant

$$\partial_\mu j^\mu(x) = 0 \tag{3.29}$$

provided the object [which involves the *adjoint spinor* $\bar{\psi}(x)$]

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) \quad \text{with} \quad \bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0 = \begin{pmatrix} \psi_1^*(x) & \ldots & \psi_4^*(x) \end{pmatrix} \gamma^0 \tag{3.30}$$

transforms like a contravariant four-vector. The adjoint spinor is always a row-vector, and specifically in the Dirac-Pauli representation (3.7) it takes the form

$$\bar{\psi}(x) \overset{\text{DP rep.}}{=} \begin{pmatrix} \psi_1^* & \psi_2^* & \psi_3^* & \psi_4^* \end{pmatrix} \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix} = \begin{pmatrix} \psi_1^*(x) & \psi_2^*(x) & -\psi_3^*(x) & -\psi_4^*(x) \end{pmatrix} \tag{3.31}$$

while in other representations it looks more complicated (since then $\gamma^0$ is not diagonal).

## 3.6 Free-field solutions

It is straight-forward to Fourier transform the Dirac equation (3.23). With the definition (2.28) it follows that $\partial_t = -\mathrm{i}\omega = -\mathrm{i}E$ and $\partial_{\vec{x}} \equiv \vec{\nabla} = \mathrm{i}\vec{k} = \mathrm{i}\vec{p}$, where $\hbar = 1$ is adopted. Hence

$$\boxed{(\gamma^\mu \mathrm{i}\partial_\mu - m)\psi(x) = 0} \qquad \Longleftrightarrow \qquad \boxed{(\gamma^\mu p_\mu - m)\hat{\psi}(p) = 0} \tag{3.32}$$

where $p^\bullet = (p^\mu) = (E, \vec{p})$ or $p_\bullet = (p_\mu) = (E, -\vec{p})$ has been used. It is common practice to consider $\psi(x) = \psi(t, \vec{x})$ for one fixed four-momentum

$$\psi(t, \vec{x}) = u(E, \vec{p})\, e^{\mathrm{i}[\vec{p}\vec{x} - Et]} = u(E, \vec{p})\, e^{-\mathrm{i}p^\mu x_\mu} = u(E, \vec{p})\, e^{-\mathrm{i}p_\mu x^\mu} \tag{3.33}$$

so that the momentum spinor $u$ is the Fourier transform of $\psi$ evaluated at that $E$ and $\vec{p}$. With this convention the Dirac equation (3.32) can be written as $(\gamma^\mu p_\mu - m)u(E, \vec{p}) = 0$.

To find solutions to (3.32) let us first consider spinors at rest, $\vec{p} = \vec{0}$. Then $E\gamma^0 u = mu$ means $E\mathrm{diag}(1, 1, -1, -1)u = mu$, and this has four independent solutions

$$\underbrace{\underbrace{u_1(E,\vec{0})}_{S_z=+\frac{1}{2}} = N \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \;,\; \underbrace{u_2(E,\vec{0})}_{S_z=-\frac{1}{2}} = N \begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{E>0} \;,\; \underbrace{\underbrace{u_3(E,\vec{0})}_{S_z=+\frac{1}{2}} = N \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \;,\; \underbrace{u_4(E,\vec{0})}_{S_z=-\frac{1}{2}} = N \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{E<0}$$

where $u_1, u_2$ work only for $E > 0$, and $u_3, u_4$ for only for $E < 0$. Verify that $S_z = \frac{1}{2} I_2 \otimes \sigma_z$ acts as indicated. Note that the $i$ in $u_i$ is not an index but part of the name of the spinor.

Now for solutions with $\vec{p} \neq \vec{0}$. With $\vec{p} \cdot \vec{x} = p_x = p^1 = -p_1$ the Dirac equation reads

$$\Big(\gamma^0 E - [\gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z] - m\Big)u = 0 \tag{3.34}$$

with $\gamma^0 E = \mathrm{diag}(E, E, -E, -E)$, $m = \mathrm{diag}(m, m, m, m)$ and a non-diagonal middle term

$$[\gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z] = \begin{pmatrix} 0 & \vec{\sigma}\vec{p} \\ -\vec{\sigma}\vec{p} & 0 \end{pmatrix} \;, \quad \text{where} \quad \vec{\sigma}\vec{p} = \begin{pmatrix} p_z & p_x - \mathrm{i}p_y \\ p_x + \mathrm{i}p_y & -p_z \end{pmatrix} \;. \tag{3.35}$$

Denoting the two "upper" and "lower" components $u_{\mathrm{upp}}$ and $u_{\mathrm{low}}$, respectively, we have

$$\begin{pmatrix} (E-m)I & -\vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & -(E+m)I \end{pmatrix} \begin{pmatrix} u_{\mathrm{upp}} \\ u_{\mathrm{low}} \end{pmatrix} = 0 \tag{3.36}$$

or $(E - m)u_{\mathrm{upp}} = (\vec{\sigma}\vec{p})u_{\mathrm{low}}$ and $(\vec{\sigma}\vec{p})u_{\mathrm{upp}} = (E + m)u_{\mathrm{low}}$ with $\vec{\sigma}\vec{p}$ given above. Hence, defining $u_{\mathrm{low}}$ to be a (2-dimensional) unit-vector specifies $u_{\mathrm{upp}}$, and vice versa. In explicit terms

$$u_{\mathrm{low}} = \left\{ \begin{pmatrix}1\\0\end{pmatrix} , \begin{pmatrix}0\\1\end{pmatrix} \right\} \qquad \longrightarrow \qquad u_{\mathrm{upp}} = \frac{\vec{\sigma}\vec{p}}{E-m}\left\{ \begin{pmatrix}1\\0\end{pmatrix} , \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

$$u_{\mathrm{upp}} = \left\{ \begin{pmatrix}1\\0\end{pmatrix} , \begin{pmatrix}0\\1\end{pmatrix} \right\} \qquad \longrightarrow \qquad u_{\mathrm{low}} = \frac{\vec{\sigma}\vec{p}}{E+m}\left\{ \begin{pmatrix}1\\0\end{pmatrix} , \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

and this means that a complete set of four four-spinors for a given $(E, \vec{p})$ combination is

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+\mathrm{i}p_y}{E+m} \end{pmatrix} , \quad u_2 = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-\mathrm{i}p_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} , \quad u_3 = N \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+\mathrm{i}p_y}{E-m} \\ 1 \\ 0 \end{pmatrix} , \quad u_4 = N \begin{pmatrix} \frac{p_x-\mathrm{i}p_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix} \tag{3.37}$$

with $N = \sqrt{E+m}$ to achieve the proper normalization of $2E$ particles per unit-volume. Note that none of these spinors is an eigenvector to $S_z$ (unless $\vec{p} = \vec{0}$). Note, finally, that it is impossible to interpret all four solutions as having $E > 0$, since then the exponent of the wavefunction $\psi(t, \vec{x}) = u \exp(\mathrm{i}[Et - \vec{p}\vec{x}])$ would be the same for all of them. In consequence, they would no longer be independent, for instance $u_1 = \frac{p_z}{E+m} u_3 + \frac{p_x+\mathrm{i}p_y}{E+m} u_4$. In summary:

- any of $u_{1,2,3,4}(E, \vec{p})$ satisfies the Dirac equation and thus $E^2 - \vec{p}^2 = m^2$
- $u_1, u_2$ have $E = +\sqrt{\vec{p}^2 + m^2} > 0$ and two spin components (but $S_z = \pm\frac{1}{2}$ only for $\vec{p} \to \vec{0}$)
- $u_3, u_4$ have $E = -\sqrt{\vec{p}^2 + m^2} < 0$ and two spin components (but $S_z = \pm\frac{1}{2}$ only for $\vec{p} \to \vec{0}$)
- problem with $\rho < 0$ is gone, the one with $E \gtrless 0$ persists; how shall we interpret $E < 0$ ?
- spinors (3.37) refer to Dirac-Pauli representation; transformed in other representations

## 3.7 Interpretation – Dirac versus Stückelberg-Feynman

Consider Fig. 4.1 in the book. Energy levels are discrete for a finite spatial extent; in infinite volume there is a continuum above $+m$ (and ditto below $-m$). The two columns for a given $E$ indicate the two $s_z$ values. The Dirac interpretation matches the standard view in solid state physics; the "vacuum" corresponds to a filled Fermi sea. A hole in the set of states with $E < 0$ behaves like a state with more energy (less negative energy) than the "vacuum" and quantum numbers opposite to what is needed to fill the hole. Accordingly, a photon that creates a hole at $E = -m_e$ along with an electron at $E = +m_e$ needs *more* energy than $2m_e[c^2]$, the surplus going into kinetic energy of the $e^+e^-$ pair. This, together with the experimental discovery of the positron (Anderson, 1933) created a triumph for Dirac. However, there is a conceptual problem. The Dirac interpretation rests on the Pauli principle; it works only for fermions. But in experiment antiparticles are found to exist for bosons, too.

In the previous subsection we stressed that we cannot change the phase factor, but

$$\exp(\mathrm{i}[Et - \vec{p}\vec{x}]) = \exp(\mathrm{i}[(-E)(-t) - \vec{p}\vec{x}])$$

suggests that we can flip the sign of $E$ if we also change the arrow of time. The modern interpretation, due to Stückelberg and Feynman, is that the $E < 0$ solutions to the Dirac equation are seen as *negative energy particles* which move *backward in time*. Due to the (trivial) mathematical identity listed above, this is equivalent to physical *positive energy antiparticles* moving *forward in time*. The difference is illustrated in Fig. 4.2 in the book (time increases from the left to the right, the $y$-axis is topology not space-time). In the left panel a positive energy electron produces a photon and thus converts itself to a negative energy electron moving backward in time (Dirac view). Here the arrows show the flow of the quantum number associated with the state indicated next to the line (hence of the $e^-$ in both cases). In the right panel a positive energy electron annihilates with a positive energy positron to produce a photon with energy $> 2m_e[c^2] > 0$ (Stückelberg-Feynman view). Again the arrows show the flow of the quantum number associated with the state indicated next to the line (one $e^-$, one $e^+$ charge).

In the remainder of this course we will use the modern Stückelberg-Feynman interpretation. Thus we shall use Feynman diagrams as in the right panel of Fig. 4.2 of the book. But we (and the book) will change the convention for the direction of the arrows. The arrows will always refer to the quantum numbers associated with the *field* (in QFT) that is used to describe both the particle and the antiparticle. Since in QED the electron-field $\psi(x)$ describes both the electrons and the positrons, the arrows will always indicate the flow of the $e^-$ charge.

In principle, one could perform all calculations with the spinors $u_1, u_2, u_3, u_4$. But one would need to keep in mind that the latter two come with a phase factor opposite to the factor $e^{\mathrm{i}[\vec{p}\vec{x}-Et]}$ used for $u_1, u_2$. In addition, $u_3, u_4$ are labeled by an $E$ which is *minus* their energy and a $\vec{p}$ which is *minus* their physical momentum. The convenient solution is to keep $u_1, u_2$ with the phase factor $e^{\mathrm{i}[\vec{p}\vec{x}-Et]}$ unchanged. But $u_3, u_4$ are traded for $v_1, v_2$ under the terms

$$v_1(E,\vec{p})e^{-\mathrm{i}[\vec{p}\vec{x}-Et]} \equiv u_4(-E,-\vec{p})e^{\mathrm{i}[-\vec{p}\vec{x}-(-E)t]}$$
$$v_2(E,\vec{p})e^{-\mathrm{i}[\vec{p}\vec{x}-Et]} \equiv u_3(-E,-\vec{p})e^{\mathrm{i}[-\vec{p}\vec{x}-(-E)t]}$$

which means that we consistently substitute $E \to -E$ and $\vec{p} \to -\vec{p}$ both in the spinor and the phase factor (for these two guys). Hence we end up (again) having four independent states

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+\mathrm{i}p_y}{E+m} \end{pmatrix} \;,\quad u_2 = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-\mathrm{i}p_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \;,\quad v_1 = N\begin{pmatrix} \frac{p_x-\mathrm{i}p_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \;,\quad v_2 = N\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+\mathrm{i}p_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \tag{3.38}$$

each one to be used with $E > 0$, and $N = \sqrt{E+m}$ to achieve the proper normalization of $2E$ particles per unit-volume. Still, none of these spinors is an eigenvector to $S_z$ (unless $\vec{p} = \vec{0}$). In short the Stückelberg-Feynman reinterpretation yields the following paradigm:

- states $u_1$, $u_2$ represent positive energy particles, with $\psi(t,\vec{x}) = u(E,\vec{p})e^{\mathrm{i}[\vec{p}\vec{x}-Et]}$
- states $v_1$, $v_2$ represent positive energy antiparticles, with $\psi(t,\vec{x}) = v(E,\vec{p})e^{-\mathrm{i}[\vec{p}\vec{x}-Et]}$
- all spinors $u_1, u_2, v_1, v_2$ are labeled by their physical energy $E > 0$ and physical momentum $\vec{p}$
- throughout $t$ increases; in Feynman diagrams $t$ flows from the left to the right
- spinors (3.38) refer to Dirac-Pauli representation; transformed in other representations

## 3.8 Charge conjugation, parity, spin and helicity

There are three discrete transformations, charge $C$, parity $P$, and time reversal $T$. An interaction may violate any of $C, P, T$, but in a local relativistic QFT the product $CPT$ (in any order) must be respected ("CPT theorem"). Here we discuss how $C$ and $P$ act on Dirac spinors.

In quantum mechanics the coupling of a particle with charge $q$ to an electromagnetic field is implemented by the minimal substitution $E \to E - q\phi$ and $\vec{p} \to \vec{p} - q\vec{A}$, with $\phi(t,\vec{x})$ and $\vec{A}(t,\vec{x})$ the scalar and vector potential, respectively. Given $p^\bullet = (E,\vec{p})$ and $A^\bullet = (\phi,\vec{A})$ we can write this as $p^\mu \to p^\mu - qA^\mu$ or $p_\mu \to p_\mu - qA_\mu$. With the phase factor $e^{-\mathrm{i}[Et-\vec{p}\vec{x}]} = e^{-\mathrm{i}px} = e^{-\mathrm{i}p_\mu x^\mu}$ of (2.28) we have $\partial_\mu = -\mathrm{i}p_\mu$ or $p_\mu = \mathrm{i}\partial_\mu$. The minimal substitution in covariant language is thus

$$\mathrm{i}\partial_\mu \to \mathrm{i}\partial_\mu - qA_\mu \qquad \Longleftrightarrow \qquad p_\mu \to p_\mu - qA_\mu \tag{3.39}$$

and this means that the free-field Dirac equation (3.32) takes the more general form

$$\boxed{(\gamma^\mu \mathrm{i}\partial_\mu - q\gamma^\mu A_\mu - m)\psi(x) = 0} \qquad \Longleftrightarrow \qquad \boxed{(\gamma^\mu p_\mu - q\gamma^\mu A_\mu - m)\hat{\psi}(p) = 0} \tag{3.40}$$

if the particle described by the field $\psi$ has charge $q$. In particle physics it is common practice, whenever $\psi$ describes the 1st-generation lepton field, to consider the $e^-$ the particle and the $e^+$ the antiparticle. However, there are two conventions for $e$, for some it is the positron charge ($e > 0$), for some it is the electron charge ($e < 0$). The book follows the first convention, so the charge of the electron is $-e$. In this case the Dirac equation for the electron field reads

$$\boxed{(\gamma^\mu[\mathrm{i}\partial_\mu + eA_\mu] - m)\psi(x) = 0} \quad \text{or} \quad \boxed{(\gamma^\mu[\partial_\mu - \mathrm{i}eA_\mu] + \mathrm{i}m)\psi(x) = 0}\,. \tag{3.41}$$

The charge operation acts on the Dirac field $\psi$, and hence on the spinors $u_{1,2}, v_{1,2}$, as

$$C: \qquad \psi(x) \longmapsto \psi'(x) \equiv C[\psi(x)] \equiv \mathrm{i}\gamma^2\psi^*(x) \tag{3.42}$$

and this means that $C$ is an involution, i.e. $C^2 = 1$. That $\gamma^2$ is used in conjunction with complex conjugation is not so surprising, since in the DP-representation it is the only one which changes sign under complex conjugation, $(\gamma^0)^* = \gamma^0, (\gamma^1)^* = \gamma^1, (\gamma^2)^* = -\gamma^2, (\gamma^3)^* = \gamma^3$. For $\psi(t,\vec{x}) = u_1(E,\vec{p})e^{-\mathrm{i}[Et-\vec{p}\vec{x}]}$ the transformed wavefunction is $\psi' = C\psi = \mathrm{i}\gamma^2 u_1^*(E,\vec{p})e^{\mathrm{i}[Et-\vec{p}\vec{x}]}$. For the spinor part we invoke the explicit form of $\gamma^2$ in the DP-representation to find

$$\mathrm{i}\gamma^2 u_1^* = \mathrm{i}\begin{pmatrix} & & & -\mathrm{i} \\ & & \mathrm{i} & \\ & \mathrm{i} & & \\ -\mathrm{i} & & & \end{pmatrix} \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+\mathrm{i}p_y}{E+m} \end{pmatrix}^* = \sqrt{E+m}\begin{pmatrix} \frac{p_x-\mathrm{i}p_y}{E+m} \\ \frac{-p_z}{E+m} \\ -0 \\ 1 \end{pmatrix} = v_1$$

$$\mathrm{i}\gamma^2 u_2^* = \mathrm{i}\begin{pmatrix} & & & -\mathrm{i} \\ & & \mathrm{i} & \\ & \mathrm{i} & & \\ -\mathrm{i} & & & \end{pmatrix} \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-\mathrm{i}p_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}^* = \sqrt{E+m}\begin{pmatrix} \frac{-p_z}{E+m} \\ -\frac{p_x+\mathrm{i}p_y}{E+m} \\ -1 \\ 0 \end{pmatrix} = -v_2$$

where the last minus sign would be absent if we would have defined $v_2$ in (3.38) differently. Hence, ignoring this sign issue, the action of $C$ can be summarized as

$$u_1 e^{-\mathrm{i}[Et-\vec{p}\vec{x}]} \quad \overset{C}{\longleftrightarrow} \quad v_1 e^{\mathrm{i}[Et-\vec{p}\vec{x}]} \tag{3.43}$$

$$u_2 e^{-\mathrm{i}[Et-\vec{p}\vec{x}]} \quad \overset{C}{\longleftrightarrow} \quad v_2 e^{\mathrm{i}[Et-\vec{p}\vec{x}]} \tag{3.44}$$

where the complex conjugation turned the phase factors around. Hence the four components stand four two spin orientations [in a complicated form, see below], and the particle/antiparticle content, where the latter relation is not so complicated, since (3.38) provides a convenient basis.

An important point to keep in mind is that the operations which give the *physical energy* and the *physical momentum* of an antiparticle are not $H_\mathrm{D}$ and $\vec{P}$, respectively, but $-H_\mathrm{D}$ and $-\vec{P}$. This is consistent with the antiparticle spinor carrying the opposite phase, i.e. $\psi = ve^{\mathrm{i}px} = ve^{\mathrm{i}[Et-\vec{p}\vec{x}]}$, as opposed to $\psi = ue^{-\mathrm{i}px} = ue^{-\mathrm{i}[Et-\vec{p}\vec{x}]}$ for a particle spinor. Hence, a factor $E > 0$ is pulled down by acting with $-\mathrm{i}\partial_t$ on the antiparticle, and with (the usual) $\mathrm{i}\partial_t$ on a particle. Similarly, a factor $\vec{p}$ is pulled down by acting with $\mathrm{i}\vec{\nabla}$ on the antiparticle, versus (the usual) $-\mathrm{i}\vec{\nabla}$ for a particle. This impacts the angular momentum operator; for a particle $\vec{L} = -\mathrm{i}\vec{R} \wedge \vec{\nabla}$ generates the physical angular momentum, while for an antiparticle $+\mathrm{i}\vec{R} \wedge \vec{\nabla}$ will do the job. At this point one might wonder what happens to a total angular momentum, for particles given by $\vec{J} = \vec{L} + \vec{S}$. Fortunately, for antiparticles also the *physical spin in a given direction* is given through $-\vec{S}$ rather than $\vec{S}$ (for $S^2 = \vec{S}^2$ it makes no difference anyway). Recall that

$$S_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix} \sigma_z & \\ & \sigma_z \end{pmatrix} = \frac{1}{2}\mathrm{diag}(1,-1,1,-1) \tag{3.45}$$

and this gives the result of $S_z$ on a $u$ or $v$ spinor where the object flies in the $\pm z$ direction

$$\begin{aligned} S_z u_1(E,0,0,\pm p) &= +\frac{1}{2}u_1(E,0,0,\pm p) \\ S_z u_2(E,0,0,\pm p) &= -\frac{1}{2}u_2(E,0,0,\pm p) \\ S_z v_1(E,0,0,\pm p) &= -\frac{1}{2}v_1(E,0,0,\pm p) \\ S_z v_2(E,0,0,\pm p) &= +\frac{1}{2}v_2(E,0,0,\pm p) \end{aligned} \tag{3.46}$$

but the *physical spin* in $z$-direction is the opposite of the prefactor for $v_{1,2}$. Hence, if everybody flies in the $z$-direction, $u_1$ is spin-up, $u_2$ is spin-down, $v_1$ is spin-up, $v_2$ is spin-down. If everybody flies in the $-z$-direction, $u_1$ is spin-up, $u_2$ is spin-down, $v_1$ is spin-up, $v_2$ is spin-down relative to the original $+z$-direction. However, in this case it would be more useful to have the preferred direction also pointing in the $-z$-direction. Hence, relative to the direction of flight, $u_1$ is spin-down, $u_2$ is spin-up, $v_1$ is spin-down, $v_2$ is spin-up (see Fig. 4.3 in the book).

What we have introduced, with this consideration, is the concept of *helicity*

$$H \equiv \frac{\vec{S}\vec{p}}{|\vec{p}|} = \frac{\Sigma_x p_x + \Sigma_y p_y + \Sigma_z p_z}{2|\vec{p}|} = \frac{1}{2|\vec{p}|}\begin{pmatrix} \vec{\sigma}\vec{p} & \\ & \vec{\sigma}\vec{p} \end{pmatrix} \tag{3.47}$$

or spin in the direction of flight (it replaces $S_z$, but not $S^2$). This is useful, because (unlike in non-relativistic QM) the operator $S_z$ is *not a constant of motion*. The relations

$$\boxed{[H_\mathrm{D}, S^2] = 0\,, \quad [S^2, S_z] = 0\,, \quad [S_z, H_\mathrm{D}] \neq 0}$$
$$\boxed{[H_\mathrm{D}, S^2] = 0\,, \quad [S^2, H\,] = 0\,, \quad [H, H_\mathrm{D}] = 0} \tag{3.48}$$

tell us that it makes sense to select $\{H_\mathrm{D}, S^2, H\}$ as our *maximal set of compatible operators*. However, there is a price to pay, since the resulting helicity $h$ is *not a Lorentz-invariant quantity*.

An explicit form of $u_{1,2}, v_{1,2}$ follows from requesting that it is an eigenstate of $H$:

$$\frac{1}{2p}\begin{pmatrix} \vec{\sigma}\vec{p} & \\ & \vec{\sigma}\vec{p} \end{pmatrix}\begin{pmatrix} u_\mathrm{upp} \\ u_\mathrm{low} \end{pmatrix} \overset{(!)}{=} \lambda \begin{pmatrix} u_\mathrm{upp} \\ u_\mathrm{low} \end{pmatrix}$$

$$\frac{1}{4p^2}(\vec{\sigma}\vec{p})^2 u_\mathrm{upp} = \lambda^2 u_\mathrm{upp} \quad \text{with} \quad (\vec{\sigma}\vec{p})^2 = \vec{p}^2 = p^2 \qquad \longrightarrow \qquad \lambda = \pm\frac{1}{2}$$

$$(\vec{\sigma}\vec{p})u_\mathrm{upp} = (E+m)u_\mathrm{low} \quad \longrightarrow \quad u_\mathrm{low} = 2\lambda\frac{p}{E+m}u_\mathrm{upp}$$

$$\vec{p} = p\begin{pmatrix} \sin\theta\,\cos\varphi \\ \sin\theta\,\sin\varphi \\ \cos\theta \end{pmatrix} \quad \longrightarrow \quad \frac{1}{2p}(\vec{\sigma}\vec{p}) = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-\mathrm{i}\varphi} \\ \sin\theta\, e^{\mathrm{i}\varphi} & -\cos\theta \end{pmatrix}$$

$$u_{\rm upp} = \begin{pmatrix} a \\ b \end{pmatrix} \quad\longrightarrow\quad \begin{pmatrix} \cos\theta & \sin\theta\, e^{-\mathrm{i}\varphi} \\ \sin\theta\, e^{\mathrm{i}\varphi} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \overset{(!)}{=} 2\lambda \begin{pmatrix} a \\ b \end{pmatrix} \quad\longrightarrow\quad \frac{b}{a} = \frac{2\lambda - \cos\theta}{\sin\theta} e^{\mathrm{i}\varphi}$$

$$\text{case } \lambda = +\frac{1}{2} \quad\longrightarrow\quad \frac{b}{a} = \frac{+1-\cos\theta}{\sin\theta} e^{\mathrm{i}\varphi} = \frac{+2s^2}{2sc} e^{\mathrm{i}\varphi} = +\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} e^{\mathrm{i}\varphi} = +\tan\frac{\theta}{2} e^{\mathrm{i}\varphi}$$

$$\text{case } \lambda = -\frac{1}{2} \quad\longrightarrow\quad \frac{b}{a} = \frac{-1-\cos\theta}{\sin\theta} e^{\mathrm{i}\varphi} = \frac{-2c^2}{2sc} e^{\mathrm{i}\varphi} = -\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} e^{\mathrm{i}\varphi} = -\cot\frac{\theta}{2} e^{\mathrm{i}\varphi}$$

Hence, if an object has physical momentum $\vec{p} = p\vec{e}$, with $\vec{e}$ given by $\theta$ measured from the "north pole" and $\varphi$, defining $c \equiv \cos\frac{\theta}{2}$ and $s \equiv \sin\frac{\theta}{2}$ lets the spinors $u_{1,2}, v_{1,2}$ take the form

$$u_\uparrow \propto \begin{pmatrix} c \\ s e^{\mathrm{i}\varphi} \\ \frac{p}{E+m} c \\ \frac{p}{E+m} s e^{\mathrm{i}\varphi} \end{pmatrix}, \quad u_\downarrow \propto \begin{pmatrix} -s \\ c e^{\mathrm{i}\varphi} \\ \frac{p}{E+m} s \\ -\frac{p}{E+m} c e^{\mathrm{i}\varphi} \end{pmatrix}, \quad v_\uparrow \propto \begin{pmatrix} \frac{p}{E+m} s \\ -\frac{p}{E+m} c e^{\mathrm{i}\varphi} \\ -s \\ c e^{\mathrm{i}\varphi} \end{pmatrix}, \quad v_\downarrow \propto \begin{pmatrix} \frac{p}{E+m} c \\ \frac{p}{E+m} s e^{\mathrm{i}\varphi} \\ c \\ s e^{\mathrm{i}\varphi} \end{pmatrix} \tag{3.49}$$

where we omitted the factor $\sqrt{E+m}$ which ensures $2E$ particles/antiparticles per unit volume. People say $u_\uparrow$ and $v_\uparrow$ have *right-handed helicity* (i.e. in the direction of flight), while $u_\downarrow$ and $v_\downarrow$ have *left-handed helicity* (i.e. against the direction of flight). Later we shall introduce the notion of "right-handed chirality" and "left-handed chirality". To avoid confusion it would be better to speak of "forward helicity" and "backward helicity", but this is not commonly accepted.

The parity operation acts on the Dirac field $\psi$, and hence on the spinors $u_{1,2}, v_{1,2}$, as

$$P: \qquad \psi(x) \longmapsto \psi'(x') \equiv P[\psi(x)] \equiv \gamma^0 \psi(x) \tag{3.50}$$

and this means that $P$ is an involution, i.e. $P^2 = 1$. That $\gamma^0$ is used (without any complex conjugation) is not a surprise either, since in the DP-representation it is the only one which is hermitean and diagonal. The action on the four spinors at rest is

$$\begin{aligned} Pu_1 &= \gamma^0 u_1 = \mathrm{diag}(1,1,-1,-1)\sqrt{2m}(1,0,0,0)^{\rm t} = \sqrt{2m}(+1,0,0,0)^{\rm t} = +u_1 \\ Pu_2 &= \gamma^0 u_2 = \mathrm{diag}(1,1,-1,-1)\sqrt{2m}(0,1,0,0)^{\rm t} = \sqrt{2m}(0,+1,0,0)^{\rm t} = +u_2 \\ P\,v_1 &= \gamma^0 v_1 = \mathrm{diag}(1,1,-1,-1)\sqrt{2m}(0,0,0,1)^{\rm t} = \sqrt{2m}(0,0,0,-1)^{\rm t} = -\,v_1 \\ P\,v_2 &= \gamma^0 v_2 = \mathrm{diag}(1,1,-1,-1)\sqrt{2m}(0,0,1,0)^{\rm t} = \sqrt{2m}(0,0,-1,0)^{\rm t} = -\,v_2 \end{aligned} \tag{3.51}$$

and as a result one finds that the intrinsic parity (for $\ell = 0$) of a fermion-antifermion pair is $P(f\bar{f}) = -1$. Perhaps you recall that the corresponding result for bosons is $P(b\bar{b}) = +1$.

## 3.9 Summary

From a theory viewpoint all sections of Chap. 4 in the book are relevant. Read them carefully, and try to solve the problems. You should remember that the 4 components of a Dirac spinor describe particle/antiparticle and spin along a preferred direction (e.g. that of flight). Details depend on the chosen representation of the Dirac-Clifford algebra of $\gamma$-matrices.

# 4 Feynman diagrams in QED

## 4.1 Time-ordered perturbation theory

We discussed Fermi's golden rule $\Gamma_{fi} = 2\pi|T_{fi}|^2\rho(E_i)$, and how the perturbative expansion

$$T_{fi} \simeq \langle f|H_{\text{int}}|i\rangle - \sum_{j\neq i} \frac{\langle f|H_{\text{int}}|j\rangle\langle j|H_{\text{int}}|i\rangle}{E_j - E_i} + \sum_{k\neq j\neq i} \frac{\langle f|H_{\text{int}}|k\rangle\langle k|H_{\text{int}}|j\rangle\langle j|H_{\text{int}}|i\rangle}{(E_k - E_j)(E_j - E_i)} - ... \tag{4.1}$$

of the transition matrix element $T_{fi}$ (from some initial state $i$ to some final state $f$) fits into the framework of non-relativistic QM. Here $H_{\text{int}}$ behaves like a classical potential, and the "particle" interacts with it once, twice, three times, and so on (see Fig. 5.1 in the book). However, there are two things which strike us as odd. First, in general scattering off a potential violates three-momentum (unless it's an "effective potential" e.g. of the sun-earth system), since the potential acts like a reservoir of momentum (energy is conserved). Second, the whole setup is not relativistically covariant (and the energy/momentum-dichotomy just underlines this).

Consider a process $a, b \to c, d$ which proceeds via an intermediate state $X$ in the $t$-channel (see Fig. 5.2 in the book). In a given reference frame, two time orderings are possible:
(a) particle $a$ disintegrates $a \to X, c$, subsequently $X, b \to d$, i.e. $X$ combines with $b$ to form $d$
(b) particle $b$ disintegrates $b \to \bar{X}, d$, subsequently $\bar{X}, a \to c$, i.e. $\bar{X}$ combines with $a$ to form $c$
A similar pair of time-ordered diagrams can be drawn if $X$ is formed in the $u$-channel (under which condition ?). But there is only one time-ordered diagram in the $s$-channel (why ?).

Quantum-mechanical reasoning suggests that the two transition amplitudes take the form

$$\begin{aligned} T_{fi}^{(a)} &= \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} = \frac{\langle d|V|X + b\rangle\langle c + X|V|a\rangle}{(E_a + E_b) - (E_c + E_X + E_b)} \\ T_{fi}^{(b)} &= \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} = \frac{\langle c|V|\bar{X} + a\rangle\langle d + \bar{X}|V|b\rangle}{(E_a + E_b) - (E_d + E_{\bar{X}} + E_a)} \end{aligned} \tag{4.2}$$

for process (a) and (b), respectively. The next step is to dress the transition amplitudes $T_{fi}$ with phase-space factors to convert them into (Lorentz) invariant matrix elements $M_{fi}$. Assuming the elementary scatterings are simple, $\langle j|V|i\rangle = \langle c + X|V|a\rangle = g_a/\sqrt{2E_a 2E_c 2E_X}$ and $\langle f|V|j\rangle = \langle d|V|X + b\rangle = g_b/\sqrt{2E_b 2E_d 2E_X}$, the invariant matrix elements take the form

$$\begin{aligned} M_{fi}^{(a)} &= \sqrt{2E_a 2E_b 2E_c 2E_d}\, T_{fi}^{(a)} = \frac{1}{2E_X} \frac{g_a g_b}{E_a - E_c - E_X} \\ M_{fi}^{(b)} &= \sqrt{2E_a 2E_b 2E_c 2E_d}\, T_{fi}^{(b)} = \frac{1}{2E_X} \frac{g_a g_b}{E_b - E_d - E_X} \end{aligned} \tag{4.3}$$

where we used $E_X = E_{\bar{X}}$, and by combining the two we arrive at the invariant matrix element

$$M_{fi} \equiv M_{fi}^{(a)} + M_{fi}^{(b)} = \frac{g_a g_b}{2E_X}\Big(\frac{1}{E_a - E_c - E_X} + \frac{1}{E_b - E_d - E_X}\Big)\,. \tag{4.4}$$

Thanks to overall energy conservation $E_b - E_d = -E_a + E_c$ this can be rewritten as

$$M_{fi} = \frac{g_a g_b}{2E_X}\Big(\frac{1}{E_a - E_c - E_X} - \frac{1}{E_a - E_c + E_X}\Big) = \frac{g_a g_b}{(E_a - E_c)^2 - E_X^2} \tag{4.5}$$

and we now invoke the relativistic dispersion relation $E_X^2 - \vec{p}_X^{\,2} = m_X^2$ (which holds for intermediate states in time-ordered diagrams but not in Feynman diagrams, see below). In process (a) we have $\vec{p}_X = \vec{p}_a - \vec{p}_c$, while in (b) we have $\vec{p}_b = -\vec{p}_X + \vec{p}_d$, and thus again $\vec{p}_X = \vec{p}_d - \vec{p}_b = \vec{p}_a - \vec{p}_c$. Hence in both processes $E_X^2 = \vec{p}_X^{\,2} + m_X^2 = (\vec{p}_a - \vec{p}_c)^2 + m_X^2$, and plugging this in yields

$$M_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - (\vec{p}_a - \vec{p}_c)^2 - m_X^2} = \frac{g_a g_b}{(p_a - p_c)^2 - m_X^2} \tag{4.6}$$

which makes it plausible that $M_{fi}$ is *Lorentz invariant* (provided $g_{a,b}$ are Lorentz invariant numbers, which they are). Using overall four-momentum conservation, $p_a - p_c = p_X = p_d - p_b$, where $q \equiv p_X$ denotes the four-momentum of the *intermediate/virtual* particle $X$, we find

$$\boxed{M_{fi} = g_b \, \frac{1}{q^2 - m_X^2} \, g_a} \tag{4.7}$$

so the invariant amplitude is the product of two vertex factors, $g_{a,b}$, times a *propagator*.

The propagator is the amplitude for $X$ to propagate from the ("lower") vertex associated with the incoming particle $a$ (and vertex factor $g_a$) to the ("upper") vertex associated with the incoming particle $b$ (and vertex factor $g_b$). Note that it generates a singularity in $M_{fi}$ if (and only if) the virtual particle $X$ is *on-shell*, that is if $q^2 = m_X^2$. Within the present line of argument (where $X$ is supposed to be on-shell) this seems questionable. In the context of Feynman diagrams the on-shell condition for internal states will disappear, while the structure (4.7) persists. It is the scalar propagator in QFT, valid for any particle without spin (e.g. a pion or a physical Higgs). Let us try to summarize the most important points:

- in time-ordered diagrams time increases from the left to the right, the vertical axis is "space"
- at vertices three-momentum is conserved, but energy is not (though it is overall conserved)
- all particles, including internal ones, satisfy the "on-shell" dispersion relation $E_X^2 - \vec{p}_X^{\,2} = m_X^2$

## 4.2 From time-ordered diagrams to Feynman diagrams

In the previous subsection we followed the basic QM principle which says that amplitudes which correspond to indistinguishable processes must be summed over. In consequence we summed over the time-orderings (a) and (b), and as a by-product we realized that $M_{fi} \equiv M_{fi}^{(a)} + M_{fi}^{(b)}$ is a Lorentz-invariant quantity.

Consider Fig. 5.5 in the book. The right panel corresponds to the sum of the two time-ordered diagrams of Fig. 5.3; the vertices are drawn *vertically displaced*, i.e. at the same time. This is a Feynman diagram, and the vertical axis represents "topology" rather than "space". In the following we shall adopt a new meaning of the arrow; it will indicate the flow of the quantum number associated with the field $\psi = e, \mu, \tau$. And the four-momenta $p_a, p_b, p_c, p_d$ are always meant to flow to the right; they have nothing to do with the arrows. In the right panel $a = c$ might be an electron, and $b = d$ might be a positron (following the rules of our new convention we would then reverse the direction of the lower arrows). Since $a, c$ are external particles (sometimes called "external legs" of the Feynman diagram), they must be on-shell, i.e. $p_a^2 = p_c^2 = m_e^2$. At the upper vertex four-momentum is conserved, i.e. $p_a = p_c + q_X$ (here a convention for the flow of $q_X$ must be chosen, and we let it flow from top to bottom). However, $X$ is off-shell, so $(p_a - p_c)^2 \neq m_X^2$. Similarly, the positron states $b, d$ are on-shell, and four-momentum is conserved at the lower vertex, too. As a result, we have $p_b^2 = p_d^2 = m_e^2$, and

$p_b + q_X = p_d$. Again $(p_b - p_d)^2 \neq m_X^2$ indicates that the intermediate state $X$ is off-shell (in fact this quantity is negative in a $t$-channel diagram). Summary for the right panel:

$$q_X^2 = (p_a - p_c)^2 = (p_b - p_d)^2 \;, \qquad q_X^2 < 0 \quad [\text{“space-like''}] \;, \qquad t - \text{channel} \tag{4.8}$$

The left panel is a Feynman diagram, too, but here the intermediate state may be a different one. Again, the arrows indicate the flow of the quantum number, and four-momenta are always meant to flow to the right. Momentum conservation at the left vertex implies $p_a + p_b = q_X$, along with $p_a^2 = p_b^2 = m_e^2$ and $q_X^2 \neq m_X^2$. Momentum conservation at the right vertex implies $q_X = p_c + p_d$, along with $p_c^2 = p_d^2 = m_e^2$ and $q_X^2 \neq m_X^2$. Summary for the left panel:

$$q_X^2 = (p_a + p_b)^2 = (p_c + p_d)^2 \;, \qquad q_X^2 > 0 \quad [\text{“time-like''}] \;, \qquad s - \text{channel} \tag{4.9}$$

Since the external states $a, b, c, d$, once specified, apply to either panel, the two Feynman amplitudes (that correspond to the two diagrams) must be summed. In the case where $a, b, c, d$ all happen to be an $e^-$, both are non-zero. In fact, in this case, there are actually four diagrams, since besides the intermediate state $X = \gamma$, also the intermediate state $Y = Z^0$ contributes (to either diagram). Suppose we consider $e^-\mu^-$ scattering, i.e. $a = c = e^-$ and $b = d = \mu^-$. In this case no intermediate state contributes to the left diagram (“$s$-channel”), while still $X = \gamma$ and $Y = Z^0$ contributes to the right diagram (“$t$-channel”). Things become more complicated, if we consider $e^-\nu_e$ scattering or $e^-\bar{\nu}_e$ scattering or $e^-\nu_\mu$ scattering or $e^-\bar{\nu}_\mu$ scattering. In each case one needs to figure whether no, one, or more states contribute to the left and/or the right diagram. In case $c = d$ and a $t$-channel diagram contributes, one must not forget the crossed $u$-channel diagram. Let us try to summarize the most important points:

- in Feynman diagrams time increases from the left to the right, the vertical axis is “topology”
- at vertices every component of four-momentum is conserved (locally, hence globally too)
- external particles satisfy the “on-shell” dispersion relation $E_j^2 - \vec{p}_j^{\,2} = m_j^2$ for $j \in \{a, b, c, d\}$
- internal states may be “off-shell”, i.e. $E_X^2 - \vec{p}_X^{\,2} \neq m_X^2$; diagram becomes resonant if “on-shell”

## 4.3 Vertices and propagators in QED

Let us consider $e^-\tau^-$ scattering, see Fig. 5.6 in the book. In this case only the $t$-channel diagram is present, and we omit the contribution from the $Z^0$, so only the $\gamma$ is exchanged. This is a very good approximation to the SM for $0 < -q^2 \ll m_Z^2$, with $q \equiv p_1 - p_3$, known as QED.

Given the discussion above, we expect the invariant amplitude to take the form

$$M_{fi} = \langle \psi_c | V | \psi_a \rangle \frac{1}{q^2 - 0} \langle \psi_d | V | \psi_b \rangle \tag{4.10}$$

where $m_\gamma^2 = 0$ is already plugged in. We expect $\langle \psi_c | V | \psi_a \rangle$ to be constructed from the $u$ spinors discussed in Sec. 3 (no antiparticle is visible, hence no $v$-spinor is needed). An extra complication is that the photon has two polarization states; hence we expect a sum over two terms of the form (4.10). Specifically, for $A_\mu(x) = \varepsilon_\mu^{(\lambda)} e^{\mathrm{i}[\vec{q}\vec{x} - \omega t]}$ we expect the polarization states $\lambda \in \{1, 2\}$. The polarization vectors $\varepsilon_\bullet^{(\lambda)}$ must be space-like and orthogonal to the propagation direction of the photon; say $\varepsilon^{(1)} = (0, 1, 0, 0)^{\mathrm{t}}$ and $\varepsilon^{(2)} = (0, 0, 1, 0)^{\mathrm{t}}$ for $\vec{q} \parallel (0, 0, 1)^{\mathrm{t}}$.

Left-multiplying the “minimally substituted” Dirac equation (3.40) with $\gamma^0$ yields

$$\mathrm{i}\frac{\partial \psi}{\partial t} + \mathrm{i}\gamma^0 \vec{\gamma} \vec{\nabla} \psi - q \gamma^0 \gamma^\mu A_\mu \psi - m \gamma^0 \psi = 0 \tag{4.11}$$

where $\vec{\gamma}\vec{\nabla} = \gamma^1\partial_x + \gamma^2\partial_y + \gamma^3\partial_z$. Comparing this with $\mathrm{i}\frac{\partial\psi}{\partial t} = H_{\mathrm{QED}}\psi$ suggests the break-up

$$H_{\mathrm{QED}} = \underbrace{-\mathrm{i}\gamma^0\vec{\gamma}\vec{\nabla} + m\gamma^0}_{H_{\mathrm{QED,free}}} + \underbrace{q\gamma^0\gamma^\mu A_\mu}_{H_{\mathrm{QED,int}}} \tag{4.12}$$

where $q$ is the charge of the field (in the present context $q = -e$ for the electron-field and the tau-field). This break-up of the QED Hamiltonian follows the logic of perturbation theory. The *free QED Hamiltonian* governs the properties of the external (electron and tau) legs, and the *interaction QED Hamiltonian* yields the properties of the vertices. Putting all these educated guesses together, we expect for the upper/lower fermion line in the Feynman diagram

$$\begin{aligned}\langle\psi_c|V|\psi_a\rangle &\longrightarrow \langle\psi(p_3)|H_{\mathrm{QED,int}}|\psi(p_1)\rangle = u^\dagger(p_3)(-e)\gamma^0\gamma^\mu\varepsilon_\mu^{(\lambda)}u(p_1)\Big|_{m\to m_e}\\ \langle\psi_d|V|\psi_b\rangle &\longrightarrow \langle\psi(p_4)|H_{\mathrm{QED,int}}|\psi(p_2)\rangle = u^\dagger(p_4)(-e)\gamma^0\gamma^\nu\varepsilon_\nu^{(\lambda)*}u(p_2)\Big|_{m\to m_\tau}\end{aligned} \tag{4.13}$$

where we already plugged in $q = -e$ for either line, and we chose different contraction indices ($\mu$ versus $\nu$) in the two vertices. Often an extra index indicates whether the spinor $u$ is meant for the $e^-$ field or the $\tau^-$ field. Since only the mass of the lepton matters, we prefer the notation above. Experimentalists decide which linear combination of $u_{1,2}$ our $u$ actually stands for.

Knowing these ingredients and the photon-propagator, we expect the invariant amplitude

$$M_{fi} = \sum_{\lambda=1,2}\Big[u^\dagger(p_3)(-e)\gamma^0\gamma^\mu u(p_1)\Big]_{m\to m_e}\frac{\varepsilon_\mu^{(\lambda)}\varepsilon_\nu^{(\lambda)*}}{q^2}\Big[u^\dagger(p_4)(-e)\gamma^0\gamma^\nu u(p_2)\Big]_{m\to m_\tau} \tag{4.14}$$

where $q$ is the four-momentum of the virtual photon (in general $q^2 \neq 0$). Based on the property $\sum_{\lambda=1,2}\varepsilon_\mu^{(\lambda)}\varepsilon_\nu^{(\lambda)*} = -\eta_{\mu\nu}$, shown in App. D of the book, and our notation $\bar{u} = u^\dagger\gamma^0$ we have

$$M_{fi} = -\Big[\bar{u}(p_3)e\gamma^\mu u(p_1)\Big]_{m\to m_e}\frac{\eta_{\mu\nu}}{q^2}\Big[\bar{u}(p_4)e\gamma^\nu u(p_2)\Big]_{m\to m_\tau}\,. \tag{4.15}$$

## 4.4 Feynman rules of QED

Feynman rules are a recipe for generating $-\mathrm{i}M_{fi}$ as a product:

| | | |
|---|---|---|
| $i$-fermion | $\longrightarrow\!\!\!\bullet$ | $u(p)$ |
| $f$-fermion | $\bullet\!\!\!\longrightarrow$ | $\bar{u}(p)$ |
| $i$-antifermion | $\longleftarrow\!\!\!\bullet$ | $\bar{v}(p)$ |
| $f$-antifermion | $\bullet\!\!\!\longleftarrow$ | $v(p)$ |
| $i$-photon | wigglyline$\bullet$ | $\epsilon_\mu(p)$ |
| $f$-photon | $\bullet$wigglyline | $\epsilon_\nu^*(p)$ |
| boson-propagator | $\bullet$wigglyline$\bullet$ | $-\mathrm{i}\frac{\eta_{\rho\sigma}}{q^2}$ |
| fermion-propagator | $\bullet\!\!\longrightarrow\!\!\bullet$ | $\frac{\mathrm{i}}{\not{q}-m}$ |
| vertex | $-\bullet-$ wiggl | $-\mathrm{i}q\gamma^\kappa$ |

Obviously, several comments are in order. First, note that there are *external momenta* (here denoted by $p$ and *internal momenta* (here denoted by $q$). In tree-level diagrams the $q$ follow from the $p$ by four-momentum conservation, in loop-digrams one ore more momentum integrations are necessary (see later). Second, the $q$ in the vertex stands for the charge of the continuous fermion line, e.g. $-e$ for a $\tau$-line, $\frac{2}{3}e$ for a U-quark $(u,c,t)$ line, $-\frac{1}{3}e$ for a D-quark $(d,s,b)$ line. Third, the "slash-notation" in the fermion-propagator is to be understood as

$$\frac{\mathrm{i}}{\not{q}-m} = \frac{\mathrm{i}}{\gamma^\mu q_\mu - m} = \frac{\mathrm{i}}{\gamma^\mu q_\mu - m}\cdot\frac{\gamma^\nu q_\nu + m}{\gamma^\nu q_\nu + m} = \frac{\mathrm{i}(\gamma^\nu q_\nu + m)}{q^2-m^2} = \frac{\mathrm{i}(\not{q}+m)}{q^2-m^2} \tag{4.16}$$

since $\gamma^\mu q_\mu \gamma^\nu q_\nu = \gamma^\mu\gamma^\nu q_\mu q_\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}q_\mu q_\nu = \eta^{\mu\nu}q_\mu q_\nu = q^2$. The slash-notation is convenient, as it avoids a potential source of confusion between a fermion propagator/leg (which has a saturated index) and the superficial Greek vertex index (which matches one of the photon-propagator indices). Finally, note that each fermion line, when read *against the flavor arrow* results in an object which transforms like a four-vector. This object is $j^\rho = \bar{u}(p_\mathrm{out})\gamma^\rho u(p_\mathrm{in})$ if the superficial index was $\rho$ and the line was a fermion, and it is $j^\sigma = \bar{v}(p_\mathrm{in})\gamma^\sigma v(p_\mathrm{out})$ if the chosen index was $\sigma$ and the line was an anti-fermion. The ordering of the in/out momenta is important; it reflects the rule "against the flavor arrow". Let us consider a few examples.

(1) Scattering of $e^-$ and $\tau^+$. Here only the $t$-channel contributes. Let the incoming $e^-$ have momentum $p_1$, the outgoing $p_3$. For the $\tau^+$ the incoming momentum is $p_2$, the outgoing $p_4$. With the superficial indices $\rho$ at the upper $(ee\gamma)$ and $\sigma$ at the lower $(\tau\tau\gamma)$ vertex, we end up, for $-\mathrm{i}M$, with $[\bar{u}(p_3)(\mathrm{i}e\gamma^\rho)u(p_1)]_{m\to m_e}(-\mathrm{i}\eta_{\rho\sigma}/q^2)[\bar{v}(p_2)(\mathrm{i}e\gamma^\sigma)v(p_4)]_{m\to m_\tau}$.

(2) Scattering of $e^-$ and $e^+$. With the $e^+$ inheriting the momentum assignments of the $\tau^+$, the $t$-channel amplitude is $[\bar{u}(p_3)(\mathrm{i}e\gamma^\rho)u(p_1)]_{m\to m_e}(-\mathrm{i}\eta_{\rho\sigma}/q^2)[\bar{v}(p_2)(\mathrm{i}e\gamma^\sigma)v(p_4)]_{m\to m_e}$. However, in this case there is a $s$-channel contribution, too. Labeling the left/right vertex with $\mu$ and $\nu$, respectively, we have $[\bar{v}(p_2)(\mathrm{i}e\gamma^\mu)u(p_1)]_{m\to m_e}(-\mathrm{i}\eta_{\mu\nu}/q^2)[\bar{u}(p_3)(\mathrm{i}e\gamma^\mu)v(p_4)]_{m\to m_e}$.

(3) Scattering of $e^-$ and $\tau^+$ with an extra $\mu^+\mu^-$ pair in the final state. This is almost the same situation as under (1), except that the middle part of the photon line is replaced by a muon line which bends at the two extra vertices (which we may label $\alpha,\beta$, respectively) to form the two extra particles in the final state. Hence the photon propagator $(-\mathrm{i}\eta_{\rho\sigma}/q^2)$ is replaced by $(-\mathrm{i}\eta_{\rho\alpha}/q^2)\frac{\mathrm{i}}{\not{q}-m_\mu}(-\mathrm{i}\eta_{\beta\sigma}/q^2)$. And there are the extra spinors $\bar{u}(p_5)$ and $v(p_6)$ if the extra $\mu^\pm$ have momenta $p_5,p_6$, respectively. They are to be sandwiched around the fermion propagator, whereupon the whole amplitude is seen to be a complex number.

## 4.5 Summary

In a Feynman diagram time increases from the left to the right, the vertical axis is "topology". Furthermore, complete *four-momentum is conserved* at each vertex (and thus globally). External particles are "on-shell", while internal particles are (in general) "off-shell".

In the vertex factor $-\mathrm{i}q\gamma^\nu$ the $q$ represents the charge of the *fermion field* that goes through the vertex, regardless whether the legs are incoming, outgoing, or mixed.

Each vertex is given a "superficial" Greek index, say $\alpha$, and this index appears both in the gamma-matrix of the vertex, say $-\mathrm{i}q\gamma^\alpha$, and as one index of the photon propagator that attaches to this vertex, say $(-\mathrm{i}\eta_{\alpha\,\mathrm{otherindex}}/q^2)$.

Each Feynman diagram is already the sum over all possible time-orderings, all helicities of internal fermions and all polarization states of internal photons. It is specific, however, for the spin/polarization states of the incoming/outgoing states. Depending on the experimental situation, the latter must be averaged/summed over (see later).

# 5 Helicity and chirality in QED

## 5.1 Perturbation theory and loop expansion

If all internal momenta of a diagram are specified by the momenta of the external legs (be they fermion legs $u, u^\dagger, v, v^\dagger$ or photon legs $\epsilon, \epsilon^*$), the diagram is called a "tree-level" diagram. However, there are diagrams where some internal momenta are not fully constrained, because there is some internal loop where an arbitrary four-momentum $q$ may flow. The number of such loops establishes a hierarchy which is matched by the power of the fine-structure constant $\alpha_{\rm em}$ in the amplitude. This is an important theorem in QFT.

For us this is beyond reach, and this is why we proceed "by example". Take a look at Fig. 6.1 in the book. This is the leading-order (LO) diagram for the process $e^+e^- \to \mu^+\mu^-$ in QED. The $e^-$ momentum is $p_1$, the $e^+$ momentum is $p_2$; accordingly the photon momentum is $p \equiv p_1 + p_2$. Alternatively, we could have stated that $p \equiv p_3 + p_4$, where $p_3, p_4$ are the momenta of the $\mu^-, \mu^+$. Now take a look at Fig. 6.2 in the book. These are one-loop diagrams ("photon self-energy", "vertex correction", "box diagram") which bring small corrections to the LO diagram of Fig. 6.1. They contribute at the next-to-leading (NLO) level to the same physical process. In each case there is an unknown amount of momentum in the internal loop, and there are two additional vertices, bringing an extra factor $\alpha_{\rm em}$ in the amplitude. If you look carefully you may realize that there is no self-energy correction (via an attached photon) to the electron. The reason is that all self-energy corrections to external legs will be taken care of by the process of "renormalization" (here $m_e \to m_e^{\rm ren} \equiv m_e^{\rm phys}$).

Consider the fermion loop in the photon propagator in Fig. 6.2.a. Following the Feynman rules of Sec. 4 we attach superficial indices to the two vertices, say $\alpha, \beta$. The incoming momentum would be $p \equiv p_a + p_b$ (flowing to the right). The outgoing momentum would be $p \equiv p_c + p_d$ (flowing to the right). The momentum in the upper half of the fermion loop might be $p + q$ (flowing to the right); in this case the momentum in the lower half is $-q$ (flowing to the right). There is no constraint on $q$, since four-momentum is conserved with any value of $q$. According to the rules of QM we must somehow sum (i.e. integrate) over all possible values of $q$. Hence with the fermion propagators $\mathrm{i}/(\not{p} + \not{q} - m_e)$ upstairs and $\mathrm{i}/(-\not{q} - m_e)$ downstairs we expect

$$\frac{1}{(2\pi)^4}\int d^4q \ \mathrm{Tr}\Big(\frac{\mathrm{i}}{\not{p} + \not{q} - m_e}\gamma^\alpha \frac{\mathrm{i}}{-\not{q} - m_e}\gamma^\beta\Big) \tag{5.1}$$

where the trace is needed to convert the four $4 \times 4$ matrices into something scalar. The index $\alpha$ is contracted with the photon propagator $-\mathrm{i}\frac{\eta_{\mu\alpha}}{p^2}$ [alternatively with $\varepsilon_\alpha(p)$ if it were an external/incoming photon]. The index $\beta$ is contracted with the photon propagator $-\mathrm{i}\frac{\eta_{\beta\nu}}{p^2}$ [alternatively with $\varepsilon^*_\beta(p)$ if it were an external/outgoing photon].

The structure of the LO amplitude in Fig. 6.1 of the book is $\alpha_{\rm em}M_{\rm LO}$, if $\alpha_{\rm em}$ is pulled out. The first power of the fine-structure constant $\alpha_{\rm em} = e^2/(4\pi) \simeq 1/137.04$ reflects the presence of two vertices in this Feynman diagram. Similarly, the structure of the NLO amplitude is $\alpha_{\rm em}^2(M_{\rm NLO}^{(1)} + M_{\rm NLO}^{(2)} + M_{\rm NLO}^{(3)})$, with the three contributions reflecting the three diagrams in Fig. 6.2 of the book. The rules of QM imply that the transition rate is proportional to

$$\begin{aligned} |M_{fi}|^2 &= \Big(\alpha_{\rm em}M_{\rm LO} + \alpha_{\rm em}^2\sum_{j=1,3}M_{\rm NLO}^{(j)} + \ldots\Big)\Big(\alpha_{\rm em}M_{\rm LO} + \alpha_{\rm em}^2\sum_{k=1,3}M_{\rm NLO}^{(k)} + \ldots\Big)^* \\ &= \alpha_{\rm em}^2|M_{\rm LO}|^2 + \alpha_{\rm em}^3\sum_\ell\{M_{\rm LO}M_{\rm NLO}^{(\ell)*} + M_{\rm LO}^*M_{\rm NLO}^{(\ell)}\} + \alpha_{\rm em}^4\sum_{j,k}M_{\rm NLO}^{(j)}M_{\rm NLO}^{(k)*} + \ldots \end{aligned} \tag{5.2}$$

which demonstrates the interference between tree-level and loop diagrams in physical processes. Relative to the LO contribution, a one-loop contribution is suppressed by at least a factor $\alpha_{\rm em}$, and this is why it is ignored in this course. From a conceptual viewpoint, however, such higher-order diagrams are relevant in the process of *renormalization*.

## 5.2 Process $e^+e^- \to \mu^+\mu^-$ as showcase

The process $e^+e^- \to \mu^+\mu^-$ is an excellent showcase for the added scientific value of the theory "QED". In addition, it is well accessible in experiment, since $e^+e^-$ beams can be accelerated (SLAC, LEP), and $\mu^+\mu^-$ are easily distinguished from other particles in a good detector.

The LO diagram is shown in Fig. 6.1 of the book. The Feynman rules say

$$M_{fi} = \mathrm{i}\cdot\Big[\bar{v}(p_2)\mathrm{i}e\gamma^\rho u(p_1)\Big]_{m\to m_e}\cdot\Big(-\mathrm{i}\frac{\eta_{\rho\sigma}}{(p_1+p_2)^2}\Big)\cdot\Big[\bar{u}(p_3)\mathrm{i}e\gamma^\sigma v(p_4)\Big]_{m\to m_\mu} \tag{5.3}$$

if superficial indices $\rho,\sigma$ are attached to the two vertices. With $j^\rho \equiv [\bar{v}(p_2)\mathrm{i}e\gamma^\rho u(p_1)]_{m\to m_e}$ the electron current and $k^\sigma \equiv [\bar{u}(p_3)\mathrm{i}e\gamma^\sigma v(p_4)]_{m\to m_\mu}$ the muon current it can be written as

$$M_{fi} = -\frac{e^2}{s}\eta_{\rho\sigma}j^\rho k^\sigma \tag{5.4}$$

where $s = q^2 = (p_1+p_2)^2 = (p_3+p_4)^2$ is the usual Mandelstam variable. This is the amplitude for the spin/helicity configuration (in experiment) that corresponds to our choice of $u, \bar{u}, v, \bar{v}$.

In experiment we have four helicity configurations of the incoming $e^+e^-$ pair; they are shown in Fig. 6.5 of the book. The notation in the figure (R for right-handed=forward helicity, L for left-handed=backward helicity) bears conflict with the concept of right-handed chirality versus left-handed chirality that will be introduced later. It seems better to stay with the $\uparrow\downarrow$ notation that was used in Sec. 3 of this reading help (and in Chap. 4 of the book). Similarly, there are four helicity configurations of the outgoing $\mu^+\mu^-$ pair.

For a polarized beam and a polarized target, we just select the appropriate $u_\uparrow$ or $u_\downarrow$ or $v_\uparrow$ or $v_\downarrow$ in (5.3). If the beam is polarized and the target is not, we have a fully specified $j^\rho$, but four $k^\sigma$ would contribute. In such a case QM tells us that we need to *sum* over the respective *probabilities/rates*. Hence with the "beam polarization" bp being any of $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$, we have

$$|M_{\rm bp}|^2 = |M_{{\rm bp},\uparrow\uparrow}|^2 + |M_{{\rm bp},\uparrow\downarrow}|^2 + |M_{{\rm bp},\downarrow\uparrow}|^2 + |M_{{\rm bp},\downarrow\downarrow}|^2\ . \tag{5.5}$$

Conversely, if the target is polarized but the beam is not, we have a fully specified $k^\sigma$, but four $j^\rho$ that contribute. In such a case QM tells us that we need to *average* over the respective *probabilities/rates*. Hence with the "target polarization" tp being any of $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$, we have

$$|M_{\rm tp}|^2 = \frac{1}{4}\Big\{|M_{\uparrow\uparrow,{\rm tp}}|^2 + |M_{\uparrow\downarrow,{\rm tp}}|^2 + |M_{\downarrow\uparrow,{\rm tp}}|^2 + |M_{\downarrow\downarrow,{\rm tp}}|^2\Big\}\ . \tag{5.6}$$

If both the beam and the target are unpolarized, we end up with $|M|^2 = \frac{1}{4}\{...\}$, where the braces comprise 16 terms (some of which may eventually turn out to be zero).

Let us consider ultra-relativistic kinematics. The helicity eigenstates (3.49) then become

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{\mathrm{i}\varphi} \\ c \\ se^{\mathrm{i}\varphi}\end{pmatrix},\quad u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{\mathrm{i}\varphi} \\ s \\ -ce^{\mathrm{i}\varphi}\end{pmatrix},\quad v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{\mathrm{i}\varphi} \\ -s \\ ce^{\mathrm{i}\varphi}\end{pmatrix},\quad v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{\mathrm{i}\varphi} \\ c \\ se^{\mathrm{i}\varphi}\end{pmatrix} \tag{5.7}$$

with $c = \cos(\vartheta/2), s = \sin(\vartheta/2)$ in the DP-representation. For $\sqrt{s} \gg m_\mu \, [\gg m_e]$

$$p_1 \simeq \begin{pmatrix} E \\ 0 \\ 0 \\ E \end{pmatrix}, \qquad p_2 \simeq \begin{pmatrix} E \\ 0 \\ 0 \\ -E \end{pmatrix}, \qquad p_3 \simeq \begin{pmatrix} E \\ E\sin\vartheta \\ 0 \\ E\cos\vartheta \end{pmatrix}, \qquad p_4 \simeq \begin{pmatrix} E \\ -E\sin\vartheta \\ 0 \\ -E\cos\vartheta \end{pmatrix} \tag{5.8}$$

in very good approximation in the COM frame. In other words, the $e^-$ moves parallel to $\vec{e}_z$, and the $e^+$ antiparallel, as encoded in $p_{1,2}$, respectively. And the $\mu^-, \mu^+$ get deflected by an angle $\vartheta$ (in the $xz$-plane) relative to the direction of flight to the like-charged predecessor, as encoded in $p_{3,4}$. All of this is depicted in Fig. 6.4 of the book. With these kinematics (and still in the DP-representation) the $i$-spinors and the $f$-spinors take the form

$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \; v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \; v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \tag{5.9}$$

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}, \; u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix}, \; v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix}, \; v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix} \tag{5.10}$$

respectively, where $\uparrow\downarrow$ refers to helicities and $c = \cos(\vartheta/2), s = \sin(\vartheta/2)$.

Based on this, we construct the muon current $k^\sigma$ for each one of the helicity combinations

$$\begin{aligned} k^\sigma_{\uparrow\uparrow} &= \bar{u}_\uparrow(p_3)\gamma^\sigma v_\uparrow(p_4) = \{0, 0, 0, 0\} \\ k^\sigma_{\uparrow\downarrow} &= \bar{u}_\uparrow(p_3)\gamma^\sigma v_\downarrow(p_4) = 2E\{0, -\cos\vartheta, +\mathrm{i}, \sin\vartheta\} \\ k^\sigma_{\downarrow\uparrow} &= \bar{u}_\downarrow(p_3)\gamma^\sigma v_\uparrow(p_4) = 2E\{0, -\cos\vartheta, -\mathrm{i}, \sin\vartheta\} \\ k^\sigma_{\downarrow\downarrow} &= \bar{u}_\downarrow(p_3)\gamma^\sigma v_\downarrow(p_4) = \{0, 0, 0, 0\} \end{aligned} \tag{5.11}$$

where $p_3, p_4$ denote the $\mu^-, \mu^+$ momentum, respectively. The four options in the curly bracket refer to the four options for $\sigma$ (for each $\sigma$ we have $k^\sigma \in \mathbb{C}$). Based on similar computations, we could obtain the electron current $j^\rho$ for given initial momenta $p_1, p_2$. However, the identity

$$\begin{aligned} [\bar{u}(p_3)\gamma^\kappa v(p_4)]^\dagger &= [u^\dagger\gamma^0\gamma^\kappa v]^\dagger = v^\dagger(\gamma^\kappa)^\dagger(\gamma^0)^\dagger u \\ &= v^\dagger(\gamma^0\gamma^\kappa\gamma^0)(\gamma^0)u = v^\dagger\gamma^0\gamma^\kappa u = \bar{v}(p_4)\gamma^\kappa u(p_3) \end{aligned} \tag{5.12}$$

provides exactly what is needed in order to convert the $\bar{u}\gamma v$-type current $k$ into the $\bar{v}\gamma u$-type current $j$. With this trick we can directly write down the result for the electron current

$$\begin{aligned} j^\rho_{\uparrow\uparrow} &= \bar{v}_\uparrow(p_2)\gamma^\rho u_\uparrow(p_1) = \{0, 0, 0, 0\} \\ j^\rho_{\uparrow\downarrow} &= \bar{v}_\downarrow(p_2)\gamma^\rho u_\uparrow(p_1) = 2E\{0, -1, -\mathrm{i}, 0\} \\ j^\rho_{\downarrow\uparrow} &= \bar{v}_\uparrow(p_2)\gamma^\rho u_\downarrow(p_1) = 2E\{0, -1, +\mathrm{i}, 0\} \\ j^\rho_{\downarrow\downarrow} &= \bar{v}_\downarrow(p_2)\gamma^\rho u_\downarrow(p_1) = \{0, 0, 0, 0\} \end{aligned} \tag{5.13}$$

where again the four slots in the curly bracket represent the four options for $\rho$. Note that on the right-hand-side (RHS) the first helicity arrow is for the $e^+$, while on the left-hand-side (LHS) the first helicity arrow is for the $e^-$.

Now we have all ingredients needed to combine the currents (5.11, 5.13) with formula (5.4). In total the two currents allow for 16 helicity combinations, but from the content of the curly brackets it is clear that only 4 of them are non-zero. Hence, the non-zero matrix elements are

$$\begin{aligned}
M_{\uparrow\downarrow\to\uparrow\downarrow} &= -\frac{e^2}{s}\, 2E\{0,-1,-\mathrm{i},0\}\cdot 2E\{0,-\cos\vartheta,+\mathrm{i},\sin\vartheta\} \\
M_{\uparrow\downarrow\to\downarrow\uparrow} &= -\frac{e^2}{s}\, 2E\{0,-1,-\mathrm{i},0\}\cdot 2E\{0,-\cos\vartheta,-\mathrm{i},\sin\vartheta\} \\
M_{\downarrow\uparrow\to\uparrow\downarrow} &= -\frac{e^2}{s}\, 2E\{0,-1,+\mathrm{i},0\}\cdot 2E\{0,-\cos\vartheta,+\mathrm{i},\sin\vartheta\} \\
M_{\downarrow\uparrow\to\downarrow\uparrow} &= -\frac{e^2}{s}\, 2E\{0,-1,+\mathrm{i},0\}\cdot 2E\{0,-\cos\vartheta,-\mathrm{i},\sin\vartheta\}
\end{aligned} \tag{5.14}$$

where the dot-product is to be evaluated as a relativistic (indefinite) "scalar product" [because it comes from the $\eta_{\rho\sigma}j^\rho k^\sigma$ contraction of the two fermion currents, see (5.4)]. With $4E^2 = s$ in the ultra-relativistic limit, it follows that

$$M_{\uparrow\downarrow\to\uparrow\downarrow} = -\frac{e^2}{s}s(0-\cos\vartheta-1-0) = e^2(1+\cos\vartheta) = 4\pi\alpha_{\rm em}(1+\cos\vartheta) \tag{5.15}$$

and the same result is obtained for $M_{\downarrow\uparrow\to\downarrow\uparrow}$. For the two middle amplitudes almost the same result is found, except for $1\to-1$. In summary, the 4 non-zero probabilities are

$$\begin{aligned}
|M_{\uparrow\downarrow\to\uparrow\downarrow}|^2 = |M_{\downarrow\uparrow\to\downarrow\uparrow}|^2 &= (4\pi\alpha_{\rm em})^2(1+\cos\vartheta)^2 \\
|M_{\uparrow\downarrow\to\downarrow\uparrow}|^2 = |M_{\downarrow\uparrow\to\uparrow\downarrow}|^2 &= (4\pi\alpha_{\rm em})^2(1-\cos\vartheta)^2
\end{aligned} \tag{5.16}$$

if we have polarized beams and the ability to detect the helicities of the resulting $\mu^+$ and $\mu^-$.

The remaining 12 fully polarized combinations (where there is either in the $i$-state or the $f$-state a like-orientation combination of helicity arrows) are zero. This is an example of a *selection rule* (as familiar from GETA or equivalent), here for helicities. You recall that such rules reflect the presence of *good quantum numbers* and hence the *presence of symmetries*. Hence, a natural question is which symmetry manifests itself in this result. A graphical illustration of this helicity selection rule is given in Fig. 6.6 of the book (where R,L should be read as $\uparrow,\downarrow$).

Suppose we have an unpolarized beam and a detector which is insensitive to the helicity combination of the $\mu^\pm$ pair in the final state. Averaging over the beam helicities and summing over the outgoing polarizations means that we end up with $|M_{fi}|^2 = \frac{1}{4}\{...\}$, where the braces comprise (in principle) 16 terms, see (5.6) and (5.5). In our case only 4 terms are non-zero, but the prefactor remains $\frac{1}{4}$, so the final result for unpolarized beam and unpolarized target is

$$\begin{aligned}
\langle|M_{fi}|^2\rangle &= \frac{1}{4}\Big\{|M_{\uparrow\downarrow\to\uparrow\downarrow}|^2 + |M_{\uparrow\downarrow\to\downarrow\uparrow}|^2 + |M_{\downarrow\uparrow\to\uparrow\downarrow}|^2 + |M_{\downarrow\uparrow\to\downarrow\uparrow}|^2\Big\} \\
&= \frac{e^4}{4}\Big\{2(1+\cos\vartheta)^2 + 2(1-\cos\vartheta)^2\Big\} = e^4\Big\{1+\cos^2\vartheta\Big\}\,.
\end{aligned} \tag{5.17}$$

## 5.3 Indirect conclusion potential

At the very end of Chap. 3 of the book (which we skipped) it is explained that $\langle |M_{fi}|^2 \rangle$ must be combined with a factor $\frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}$, where $p_{i,f}^*$ denotes the magnitude of the (three-)momenta in the center-of-mas (COM) frame. This gives the *differential cross section*

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{e^4}{s}\left(1+\cos^2\vartheta\right) \tag{5.18}$$

for unpolarized $e^+e^- \to \mu^+\mu^-$. It makes sense that only the Mandelstam variable $s$ shows up, since we started from the observation that in this process only the $s$-channel Feynman diagram contributes. In short, QED predicts that in this process there is a forward-backward symmetry about $\cos\vartheta = 0$, i.e. a symmetry under $\vartheta \leftrightarrow \pi - \vartheta$.

This expectation is depicted in the left panel of Fig. 6.7 in the book. The ascending dotted line shows the effect of $2(1+\cos\vartheta)^2$, the descending dotted line shows the effect of $2(1-\cos\vartheta)^2$. Our result (5.18) is shown as a full line – symmetric about $\vartheta = \pi/2$ as expected.

The experimental result is shown in the right panel of Fig. 6.7 in the book. It differs from our expectation; it looks like our QED result is superimposed with an unknown *asymmetric* contribution. The fundamental symmetry which enforces the symmetry of our QED result about $\vartheta = \pi/2$ is the discrete operation of *parity*. Hence we conclude indirectly that this so-far unknown force breaks parity. From a modern perspective all of this makes sense. The missing part is exchanges mediated by weak interactions (besides the photon in Fig. 6.1 of the book there is a $Z$-boson line which contributes in the SM). These two amplitudes must be added (before taking any absolute squares), and this will eventually result in the dotted line in the right panel of Fig. 6.7 of the book. Of course, there is no logical path that leads from the experimental result in a unique way to this theory. One must rather guess this theory (here the theory of weak interactions) correctly, e.g. with the assumption that it breaks parity *in the maximal possible way*, and test whether this candidate theory passes all experimental tests.

All of this is reminiscent of the current situation in particle physics. We have a prediction (in current research by the SM rather than by QED) which obeys some symmetries. If experiment is found to deviate from this expectation, we know there are some extra – so far unaccounted for – degrees of freedom at work. Historically, it was a long way from the observation of forward-backward asymmetry in unpolarized $e^+e^- \to \mu^+\mu^-$ differential cross section data to the modern theory of weak interactions. In current research, experiment is trying to nail down deviations from the SM in a few places (e.g. in $B$-physics). Once this is established (we are not there yet), it will likely take a long time until the underlying "new physics" is guessed correctly (and verified by checking its predictions in many other places).

Last but not least it is instructive to see what happens if we proceed from the differential cross-section (5.18) to the *total cross section*. We need to integrate over $\vartheta$ and $\varphi$, with $d\Omega = d\varphi\,\sin\vartheta\,d\vartheta = d\varphi\,d(\cos\vartheta)$. With this substitution the solid-angle integral is

$$\int (1+\cos^2\vartheta)d\Omega = \int_0^{2\pi} d\varphi \int_{-1}^{+1}(1+\cos^2\vartheta)d(\cos\vartheta) = 2\pi\left\langle x + \frac{1}{3}x^3\right\rangle_{-1}^{+1} = 2\pi\frac{8}{3} \tag{5.19}$$

and the tree-level prediction for the total $e^+e^- \to \mu^+\mu^-$ cross section takes the form

$$\sigma = \frac{4\pi\alpha_{\rm em}^2}{3s}\,. \tag{5.20}$$

In Fig. 6.8 of the book this formula is compared to experimental data – things seem to agree perfectly. How is this possible ? The point is that in the solid-angle integration we would integrate over $\vartheta$ from $-\pi$ to $\pi$. In Fig. 6.7 of the book the data are once above and once below the full (QED) line. By doing the integration we become insensitive to this difference (the areas under the two curves are nearly the same). This shows that it is important to select the right observables – those which are *sensitive* to the (suspected) effects of "new physics".

## 5.4 Chirality and chiral projectors

Helicity is a concept which is intuitively accessible, but it is not Lorentz invariant. Chirality, by contrast, is Lorentz covariant, but it has no intuitive ("thumb forward", "thumb backward") interpretation. The two concepts match each other in the ultra-relativistic limit.

Chirality is based on the definition of the $4 \times 4$ matrix

$$\boxed{\gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \doteq \sigma_1 \otimes I = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}} \tag{5.21}$$

where the definition is universal, but the form to the right of the dotted equality is specific to the DP-representation. Given the properties of $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ one may derive

$$(\gamma^5)^2 = I \;, \qquad (\gamma^5)^\dagger = \gamma^5 \;, \qquad \{\gamma^\mu, \gamma^5\} = 0 \tag{5.22}$$

and these properties are universal, i.e. valid in any representation. In the limit $E \gg m$ (and *only* in this limit) the ultra-relativistic helicity eigenstates (5.7) are also eigenstates of $\gamma^5$

$$\gamma^5 u_\uparrow = u_\uparrow \;, \quad \gamma^5 u_\downarrow = -u_\downarrow \;, \quad \gamma^5 v_\uparrow = -v_\uparrow \;, \quad \gamma^5 v_\downarrow = v_\downarrow \;. \tag{5.23}$$

Note that the minus sign is with $u_\downarrow$ and $v_\uparrow$ [please check by acting with (5.21) onto (5.7)]. Away from the ultra-relativistic limit, one can define spinors which are eigenstates of $\gamma^5$

$$\gamma^5 u_R = u_R \;, \quad \gamma^5 u_L = -u_L \;, \quad \gamma^5 v_R = -v_R \;, \quad \gamma^5 v_L = v_L \;, \tag{5.24}$$

and the subscripts $R, L$ help us distinguishing such *chirality eigenstates* [relative to $\gamma^5$] from *helicity eigenstates* [relative to (3.47)] which have $\uparrow\downarrow$ subscripts. The general solutions to the Dirac equation which are also eigenstates of $\gamma^5$ are proportional to the massless states (5.7), so

$$u_R = N \begin{pmatrix} c \\ s e^{\mathrm{i}\varphi} \\ c \\ s e^{\mathrm{i}\varphi} \end{pmatrix} , \quad u_L = N \begin{pmatrix} -s \\ c e^{\mathrm{i}\varphi} \\ s \\ -c e^{\mathrm{i}\varphi} \end{pmatrix} , \quad v_R = N \begin{pmatrix} s \\ -c e^{\mathrm{i}\varphi} \\ -s \\ c e^{\mathrm{i}\varphi} \end{pmatrix} , \quad v_L = N \begin{pmatrix} c \\ s e^{\mathrm{i}\varphi} \\ c \\ s e^{\mathrm{i}\varphi} \end{pmatrix} \tag{5.25}$$

with $N = \sqrt{E+m}$ and $c = \cos(\vartheta/2), s = \sin(\vartheta/2)$ in the DP-representation.

The properties of the matrix $\gamma^5$ imply that the operators

$$P_R \equiv \frac{1}{2}(I + \gamma^5) \;, \qquad P_L \equiv \frac{1}{2}(I - \gamma^5) \tag{5.26}$$

form a complete set of projectors, that is

$$P_R P_R = P_R \;, \quad P_R P_L = 0 \;, \quad P_L P_R = 0 \;, \quad P_L P_L = P_L \;, \quad P_R + P_L = I \;. \tag{5.27}$$

Upon acting with $P_R$ on (5.25) we find

$$P_R u_R = u_R \ , P_R u_L = 0 \ , P_R v_R = 0 \ , P_R v_L = v_L \ , \tag{5.28}$$

so $P_R$ projects out the *right-handed particle states and the left-handed anti-particle states* (both in the sense of chirality, not helicity). Similarly, upon acting with $P_L$ on (5.25) we find

$$P_L u_R = 0 \ , P_L u_L = u_L \ , P_L v_R = v_R \ , P_L v_L = 0 \ , \tag{5.29}$$

so $P_L$ projects out the *left-handed particle states and the right-handed anti-particle states.* More details on the precise relation between helicity eigenstates and chirality eigenstates are provided in the book (subsection 6.4.2).

## 5.5 Trace techniques

For theoreticians trace techniques are important, for instance the "completeness relations"

$$\sum_{i=1,2} u_i(p)\bar{u}_i(p) = \not{p} + mI \ , \quad \sum_{i=1,2} v_i(p)\bar{v}_i(p) = \not{p} - mI \tag{5.30}$$

are frequently used. Here the "slash notation" is employed, i.e. $\not{p} \equiv \gamma^\mu p_\mu = p_\mu \gamma^\mu$.

Next, trace theorems are important. It is easy to prove that the product of any odd number of $\gamma$-matrices (i.e. $\gamma^0, \gamma^1, \gamma^2, \gamma^3$, but *not* $\gamma^5$) has a vanishing trace. For even products one proves

$$\mathrm{tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu} \ , \qquad \mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4\eta^{\mu\nu}\eta^{\rho\sigma} - 4\eta^{\mu\rho}\eta^{\nu\sigma} + 4\eta^{\mu\sigma}\eta^{\nu\rho} \tag{5.31}$$

besides $\mathrm{tr}(I) = 4$, of course. Traces of 6,8,... $\gamma$-matrices are more cumbersome to deal with. Also the trace of $\gamma^5$ times a product of an odd number of $\gamma$-matrices is zero (this follows from the first statement in this paragraph). The first two examples of the trace of $\gamma^5$ times a product of an even number of $\gamma$-matrices are simple, $\mathrm{tr}(\gamma^5) = 0$ and $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$, but $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\mathrm{i}\epsilon^{\mu\nu\rho\sigma}$ is more involved, and after this complexity quickly proliferates.

## 5.6 Summary

The goal of this section was to apply the technique of Feynman rules to a real-world process, $e^+e^- \to \mu^+\mu^-$ in QED (with the Mandelstam variable $s$ above threshold for $\mu^+\mu^-$ pair production, of course). In passing we discussed the concepts of *helicity* versus *chirality*.

• In some cases we summed over *amplitudes*, in some cases we summed/averaged over *probabilities*. The difference is rooted in QM; make sure you understand it well !

• The deviation of the QED tree-level prediction (5.18) from experimental data does not disappear by including higher-order loop corrections in QED. It is rooted in QED respecting parity, while weak interactions in the SM (and thus the data) violate parity.

• The chiral projectors $P_R, P_L$, defined by recurrence to $\gamma^5$, provide a relativistically invariant replacement for helicity; unfortunately there is no hands-on interpretation. They yield a decomposition $u = c_R u_R + c_L u_L = \frac{1}{2}(1+\gamma^5)u + \frac{1}{2}(1-\gamma^5)u = P_R u + P_L u$ of any particle spinor or $v = d_L v_L + d_R v_R = \frac{1}{2}(1+\gamma^5)v + \frac{1}{2}(1-\gamma^5)v = P_R v + P_L v$ of any anti-particle spinor.

• We were forced to omit the concept of *crossing symmetry* in the invariant amplitudes $M_{fi}$. Likewise Chap. 7 and Chap. 8 of the book will be skipped, since they are primarily of interest from an experimental viewpoint.

# 6 Global SU(2) and SU(3) symmetries in flavor space

## 6.1 SU(2) group and algebra

$SU(2)$ is defined as the group of special unitary matrices $U$ with $UU^\dagger = I_2$ and $\det(U) = 1$. A matrix $M \in \mathbb{C}^{2\times 2}$ has 8 (real) parameters, hence $SU(2)$ has $8-4-1=3$ parameters.

Infinitesimal $SU(2)$ transformations: $U(\epsilon) = I + \mathrm{i}\epsilon G + O(\epsilon^2)$ with $\epsilon \in \mathbb{R}$. From $U(\epsilon)U^\dagger(\epsilon) = (I+\mathrm{i}\epsilon G)(I-\mathrm{i}\epsilon G^\dagger) = I + \mathrm{i}\epsilon(G-G^\dagger) + O(\epsilon^2)$ it follows that

$$UU^\dagger = I \qquad \Longleftrightarrow \qquad G = G^\dagger \qquad \Longleftrightarrow \qquad U^\dagger U = I \tag{6.1}$$

and the chain $[H,U]=0 \;\to\; [H,1+\mathrm{i}\epsilon G]=0 \;\to\; [H,G]=0$ (with the Hamiltonian $H$) implies

$$\frac{d}{dt}\langle G\rangle = \mathrm{i}(\hbar)\langle [H,G]\rangle = 0 \tag{6.2}$$

so $G$ is a conserved quantity or the pertinent value $g$ a "good quantum number".

Finite transformations are obtained from an infinite number of infinitesimal transformations

$$U(\alpha) \equiv \lim_{n\to\infty}\Big(1+\frac{\mathrm{i}}{n}\alpha\cdot G\Big)^n = \sum_{k=1}^{\infty}\frac{\mathrm{i}^k}{k!}(\alpha\cdot G)^k \equiv \exp(\mathrm{i}\alpha\cdot G) \tag{6.3}$$

where we introduce the matrix exponential $\exp : \mathbb{C}^{n\times n} \longrightarrow \mathbb{C}^{n\times n}$. Parameterization of $U(2)$ is

$$U = \exp(\mathrm{i}\alpha_0 + \mathrm{i}\alpha\cdot G) = \exp(\mathrm{i}[\alpha_0 I + \alpha_1 G_1 + \alpha_2 G_2 + \alpha_3 G_3]) \tag{6.4}$$

with $G_0 = I$ and $G_{1,2,3}$ hermitean (for unitarity) and traceless (to ensure $\det = 1$, due to $\log\det = \mathrm{tr}\log$) matrices. The group $SU(2)$ has 3 real parameters (i.e. $\alpha_0 = 0$). The function (6.3) maps the tangent manifold (called *Lie algebra*) to the Lie group. The dimension of this algebra (4 or 3, respectively) matches the number of parameters needed to cover the group. The basis elements of this tangent space are called *generators*, i.e. our $G$ are usually denoted $T$. The usual choice is $T_0 \equiv I$ (if needed) and $T_i \equiv \sigma_i/2$. Last but not least note that

$$U = \exp(\mathrm{i}[\alpha_0 I_2 + \frac{\alpha_1}{2}\sigma_1 + \frac{\alpha_2}{2}\sigma_2 + \frac{\alpha_3}{2}\sigma_3]) = e^{\mathrm{i}\alpha_0}\exp(\mathrm{i}[\frac{\alpha_1}{2}\sigma_1 + \frac{\alpha_2}{2}\sigma_2 + \frac{\alpha_3}{2}\sigma_3]) \tag{6.5}$$

but the last term can *not* be factored into $\exp(\mathrm{i}\frac{\alpha_1}{2}\sigma_1)\exp(\mathrm{i}\frac{\alpha_2}{2}\sigma_2)\exp(\mathrm{i}\frac{\alpha_3}{2}\sigma_3)$, see the Baker-Campbell-Hausdorff formula for details.

## 6.2 Isospin symmetry for quarks

In Sec. 1 we stated $m_u \simeq 2\,\mathrm{MeV}$ and $m_d \simeq 5\,\mathrm{MeV}$ in the $\overline{\mathrm{MS}}$ scheme at a renormalization scale $\mu = 2\,\mathrm{GeV}$. This does not suggest that $u \leftrightarrow d$ might be a good symmetry. Reality is different, because these masses have to be seen in the context of typical hadronic masses/scales. The latter are of order $4\pi F_\pi = 4\pi\, 92.4\,\mathrm{MeV} \simeq 1\,\mathrm{GeV}$ or $M_p \simeq M_n \simeq 1\,\mathrm{GeV}$. Hence the statement is that *with respect to strong interactions* the world is almost symmetric between the $u$-quark and the $d$-quark, since the two are nearly massless on a typical hadronic scale (1 GeV). This was already noted by nuclear physicists looking for a strong potential to describe the interaction between nucleons, they suggested $V_{pp}(r) = V_{pn}(r) = V_{nn}(r)$.

To exploit the $SU(2)$ symmetry we put two iso-spin related states into a column vector, i.e.

$$N = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \text{ with } p = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \text{ with } u = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.6}$$

denote a generic nucleon state or a generic light ($S = C = B = T = 0$) quark state, respectively. The two states form a *doublet* under isospin, associated with quantum numbers $i \equiv \frac{1}{2}, i_3 \equiv \pm\frac{1}{2}$, in complete (mathematical) analogy to $s \equiv \frac{1}{2}, s_3 \equiv \pm\frac{1}{2}$ for a spin-$\frac{1}{2}$ doublet. Hence we deal with the operators $I, I_3$ in exactly the same way as we did with $S, S_3$.

Physicswise an immediate (and important) prediction when considering baryon states

$$|uuu\rangle, \quad |uud\rangle, \quad |udd\rangle, \quad |ddd\rangle,$$

is that the two middle states (e.g. "proton" and "neutron", respectively) are either both bound or both unbound. So far so good. Following this principle we would also predict the $|uuu\rangle$ and $|ddd\rangle$ states to be bound, and this is not the case. Below we will see that the reason is that the $p$ and $n$ sit in an octet, not a decuplet (when strangeness is included). Furthermore, one has to keep in mind that isospin is conserved only by strong interactions, it is partly or fully broken by electromagnetic (respects $I_3$) and weak (respects nothing) interactions.

The isospin algebra for quarks follows from

$$[T_1, T_2] = \mathrm{i}T_3 \text{ (cycl)} \qquad \text{and} \qquad T^2 \equiv T_1^2 + T_2^2 + T_3^2 \tag{6.7}$$

since this means that a maximal set of commuting operators is $\{H, T^2, T_3\}$. Here we denote the isospin operators $T, T_3$ instead of $I, I_3$, but we will continue to denote the quantum numbers $i, i_3$, respectively. With this convention the eigenstates are

$$T^2\phi(i, i_3) = i(i+1)\phi(i, i_3)\ , \qquad T_3\phi(i, i_3) = i_3\phi(i, i_3) \tag{6.8}$$

and we introduce "ladder operators" $T_\pm \equiv T_1 \pm \mathrm{i}T_2$ (see Fig. 9.2 in the book) whereupon

$$T_\pm\phi(i, i_3) = \sqrt{i(i+1) - i_3(i_3 \pm 1)}\,\phi(i, i_3 \pm 1) \tag{6.9}$$

for $i_3 \pm 1 \in \{-i, ..., i\}$ and 0 otherwise. Accordingly for the $i = \frac{1}{2}$ system

$$T_+u = 0\ , \quad T_-u = d\ , \quad T_+d = u\ , \quad T_-d = 0\ . \tag{6.10}$$

## 6.3 Building diquarks and baryons with 2 flavors

Let us proceed to the combination of two quarks. Take a look at Fig. 9.3 in the book. The question is how the two guys in the middle $ud, du$ would relate to the states $\phi(0,0), \phi(1,0)$. The trick is to start from one of the extremes, say $uu = \phi(1,1)$ and to proceed with the lowering operator $T_-$. In other words, we identify $uu = \phi(\frac{1}{2},\frac{1}{2})\phi(\frac{1}{2},\frac{1}{2}) \equiv \phi(1,1)$ as well as $dd = \phi(\frac{1}{2},-\frac{1}{2})\phi(\frac{1}{2},-\frac{1}{2}) \equiv \phi(1,-1)$, and use one of the ladder operators. For instance $T_-\phi(1,+1) = \sqrt{2}\phi(1,0)$ and $T_-\{uu\} = du + ud$ yield $\phi(1,0) = \frac{1}{\sqrt{2}}(ud + du)$. Alternatively $T_+\phi(1,-1) = \sqrt{2}\phi(1,0)$ and $T_+\{dd\} = ud + du$ confirm $\phi(1,0) = \frac{1}{\sqrt{2}}(ud+du)$. And we are led to the conclusion that the guy orthogonal to $\frac{1}{\sqrt{2}}(ud+du)$ is the singlet state, hence $\phi(0,0) = \frac{1}{\sqrt{2}}(ud - du)$, where the overall sign is arbitrary. There are two checks for the singlet state

$$T_+\frac{1}{\sqrt{2}}(ud - du) = \frac{1}{\sqrt{2}}\Big(T_+\{u\}d + uT_+\{d\} - T_+\{d\}u - dT_+\{u\}\Big) = \frac{1}{\sqrt{2}}(0 + uu - uu - 0) = 0$$

$$T_-\frac{1}{\sqrt{2}}(ud-du) = \frac{1}{\sqrt{2}}\Big(T_-\{u\}d + uT_-\{d\} - T_-\{d\}u - dT_-\{u\}\Big) = \frac{1}{\sqrt{2}}(dd+0-0-dd) = 0$$

and this result is depicted in Fig. 9.4 of the book. Mathematically we have re-done what you did in Physics 3&4 and TP3, we have decomposed the product of two fundamental spin-$\frac{1}{2}$ representations into irreps. This is denoted as $\frac{1}{2}\otimes\frac{1}{2} = 1\oplus 0$ or $2\otimes 2 = 3\oplus 1$, where the first notation is in terms of spins, and the second notation is in terms of multiplicities.

We are now ready to combine three quarks, i.e. to reduce out $\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}$ or $2\otimes 2\otimes 2$. Of course we start from the previous result, i.e. we aim to perform $(1\oplus 0)\otimes\frac{1}{2} = (1\otimes\frac{1}{2})\oplus(0\otimes\frac{1}{2})$ in the first notation or $(3\oplus 1)\otimes 2 = (3\otimes 2)\oplus(1\otimes 2)$ in the second notation. The second term is obviously a (iso)spin-$\frac{1}{2}$ doublet, but what is $1\otimes\frac{1}{2}$ or $3\otimes 2$ ? The in-concise view is given in Fig. 9.5. There is a linear combination of $ddu$ and $\frac{1}{2}(ud+du)d$ that will be in the $i=\frac{3}{2}$ quartet, and one orthogonal state. Likewise, there is a linear combination of $uud$ and $\frac{1}{2}(ud+du)u$ that will be in the $i=\frac{3}{2}$ quartet, and one orthogonal state. Again, the trick is to start at either $i_3$-extreme, and to use the respective ladder operator. From

$$\sqrt{3}\phi(\frac{3}{2},-\frac{1}{2}) = T_+\phi(\frac{3}{2},-\frac{3}{2}) = T_+\{ddd\} = T_+\{d\}dd + dT_+\{d\}d + ddT_+\{d\} = udd + dud + ddu$$

we conclude that we have a (iso)spin $\frac{3}{2}$ quartet/quadruplet

$$\begin{aligned}
\phi(\frac{3}{2},+\frac{3}{2}) &= uuu \\
\phi(\frac{3}{2},+\frac{1}{2}) &= \frac{1}{\sqrt{3}}(uud+udu+duu) \\
\phi(\frac{3}{2},-\frac{1}{2}) &= \frac{1}{\sqrt{3}}(ddu+dud+udd) \\
\phi(\frac{3}{2},-\frac{3}{2}) &= ddd
\end{aligned} \tag{6.11}$$

which is *totally symmetric* under any exchange of two flavors. An orthogonality request yields

$$\begin{aligned}
\phi_S(\frac{1}{2},+\frac{1}{2}) &= +\frac{1}{\sqrt{6}}(2uud-udu-duu) \\
\phi_S(\frac{1}{2},-\frac{1}{2}) &= -\frac{1}{\sqrt{6}}(2ddu-dud-udd)
\end{aligned} \tag{6.12}$$

that is a *mixed symmetric* (iso)spin $\frac{1}{2}$ doublet. The descendants of the (iso)singlet are

$$\begin{aligned}
\phi_A(\frac{1}{2},+\frac{1}{2}) &= \frac{1}{\sqrt{2}}(udu-duu) \\
\phi_A(\frac{1}{2},-\frac{1}{2}) &= \frac{1}{\sqrt{2}}(udd-dud)
\end{aligned} \tag{6.13}$$

that is a *mixed anti-symmetric* (iso)spin $\frac{1}{2}$ doublet. In total we have $2\otimes 2\otimes 2 = 4\oplus 2\oplus 2$ in the plet-notation, where 4 is totally symmetric, one 2 is symmetric among the first two entries, and the other 2 is anti-symmetric among the first two slots (for both doublets there is no definite pattern under $1\leftrightarrow 3$ or $2\leftrightarrow 3$). In spin-notation we found $\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = \frac{3}{2}\oplus\frac{1}{2}\oplus\frac{1}{2}$.

You could/should have discussed all of this in terms of combining three spin-$\frac{1}{2}$ states in your quantum mechanics course. The result is a carbon-copy of what you obtained there, i.e.

$$\begin{aligned}
\chi(\frac{3}{2},+\frac{3}{2}) &= \uparrow\uparrow\uparrow \\
\chi(\frac{3}{2},+\frac{1}{2}) &= \frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow+\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow) \\
\chi(\frac{3}{2},-\frac{1}{2}) &= \frac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow+\downarrow\uparrow\downarrow+\uparrow\downarrow\downarrow) \\
\chi(\frac{3}{2},-\frac{3}{2}) &= \downarrow\downarrow\downarrow \\
\chi_S(\frac{1}{2},+\frac{1}{2}) &= +\frac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow) \\
\chi_S(\frac{1}{2},-\frac{1}{2}) &= -\frac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow-\downarrow\uparrow\downarrow-\uparrow\downarrow\downarrow) \\
\chi_A(\frac{1}{2},+\frac{1}{2}) &= \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow) \\
\chi_A(\frac{1}{2},-\frac{1}{2}) &= \frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow-\downarrow\uparrow\downarrow)
\end{aligned} \tag{6.14}$$

is a list of all spin-states (in the spin notation) that three spin-$\frac{1}{2}$ particles may form. The symmetry pattern is again "totally symmetric", "mixed symmetric" (among $1\leftrightarrow 2$) and "mixed anti-symmetric" (among $1\leftrightarrow 2$).

With these two ingredients we can construct baryon wave-functions. The point is that

$$\psi \equiv \phi_{\text{flavor}}\cdot\chi_{\text{spin}}\cdot\xi_{\text{color}}\cdot\eta_{\text{space}} \tag{6.15}$$

must be *totally anti-symmetric* under *any complete exchange* of two slots. Here, "complete" means in flavor space, spin space, color space, and $\mathbb{R}^4$. We have discussed the options for $\phi_{\text{flavor}}$ and $\chi_{\text{spin}}$. A preview to the next section tells us that $\xi_{\text{color}}$ is totally antisymmetric by itself. And from GETA (or equivalent) you know that $\eta_{\text{space}}$ is typically $(-1)^\ell$, hence symmetric for an $S$-state. Assuming that most baryons are in their groundstate, we reach the conclusion that the product $\phi_{\text{flavor}}\cdot\chi_{\text{spin}}$ must be *totally symmetric*. How can we achieve this ?

In terms of symmetry patterns of (6.11, 6.12, 6.13) and (6.14) we have $3\cdot 3=9$ options. So we just select the right ones, and we see that

$$\psi=\phi_{\frac{3}{2}}\cdot\chi_{\frac{3}{2}} \qquad \text{and} \qquad \psi=\frac{1}{\sqrt{2}}\Big(\phi_{\frac{1}{2},S}\chi_{\frac{1}{2},S}+\phi_{\frac{1}{2},A}\chi_{\frac{1}{2},A}\Big) \tag{6.16}$$

would work. The first option in (6.16) gives the Delta ("$\Delta$)" baryons, the second option gives the nucleon ("$N$") baryons, i.e. $p$ and $n$. Combining any of the lines of (6.11) with the first four lines of (6.14) one finds the wavefunctions of the four $\Delta$ states $\Delta^-,\Delta^0,\Delta^+,\Delta^{++}$ with any of $s_z\in\{\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{2}\}$. We will discuss them in a world with $s$-quarks as part of a flavor decuplet.

The second option in (6.16) gives the nucleon ($p$ or $n$) with spin-up or spin-down ($\uparrow$ or $\downarrow$), e.g.

$$\begin{aligned}
|p,\uparrow\rangle &= \frac{1}{\sqrt{2}}\Big[\phi_S(\frac{1}{2},\frac{1}{2})\chi_S(\frac{1}{2},\frac{1}{2}) + \phi_A(\frac{1}{2},\frac{1}{2})\chi_A(\frac{1}{2},\frac{1}{2})\Big] \\
&= \frac{1}{6\sqrt{2}}(2uud - udu - duu)(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + \frac{1}{2\sqrt{2}}(udu - duu)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \\
&= \frac{1}{\sqrt{18}}\Big[2u\!\uparrow u\!\uparrow d\!\downarrow - u\!\uparrow u\!\downarrow d\!\uparrow - u\!\downarrow u\!\uparrow d\!\uparrow \\
&\qquad +2u\!\uparrow d\!\downarrow u\!\uparrow - u\!\uparrow d\!\uparrow u\!\downarrow - u\!\downarrow d\!\uparrow u\!\uparrow \\
&\qquad +2d\!\downarrow u\!\uparrow u\!\uparrow - d\!\uparrow u\!\uparrow u\!\downarrow - d\!\uparrow u\!\downarrow u\!\uparrow\Big]
\end{aligned} \tag{6.17}$$

and similarly for $|p,\downarrow\rangle$, $|n,\uparrow\rangle$, and $|n,\downarrow\rangle$.

## 6.4 Isospin symmetry for antiquarks

Here comes a specialty for $SU(2)$ into play. Recall that $SU(2)$ is a mapping of the form

$$SU(2): \quad \begin{pmatrix} u \\ d \end{pmatrix} \longmapsto \begin{pmatrix} u' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} \tag{6.18}$$

or $q \mapsto q' = Uq$ with $|\alpha|^2 + |\beta|^2 = 1$. Also recall that charge was defined through (3.42) which involves a complex conjugate, hence we have the behavior

$$SU(2): \quad \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} \longmapsto \begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} = \begin{pmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} . \tag{6.19}$$

It would be stupid to define $\bar{q}$ as $\binom{\bar{u}}{\bar{d}}$, since a specialty of $SU(2)$ is that $\bar{u}, \bar{d}$ can be assembled into $\bar{q}$ in such a way that $\bar{q}$ transforms exactly like $q$. This is no longer true for other groups, e.g. $SU(3)$. Mathematicians call it "pseudoreality of $SU(2)$". If we adopt the definition

$$\bar{q} \equiv \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} = S \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} \quad \text{with} \quad S \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} = S^{-1}\bar{q} \, , \quad \begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} = S^{-1}\bar{q}' \tag{6.20}$$

and equation (6.19) can be rewritten as $S^{-1}\bar{q}' = U^* S^{-1}\bar{q}$. Hence $\bar{q}' = SU^*S^{-1}\bar{q}$, but a simple calculation reveals that $SU^*S^{-1} = ... = U$. The final result is $\bar{q}' = U\bar{q}$, that is $\bar{q}$ transforms exactly like $q$, if $\bar{q}$ is defined in the special way indicated in (6.20). As a result of this we find

$$T_+\bar{u} = -\bar{d} \, , \quad T_-\bar{u} = 0 \, , \quad T_+\bar{d} = 0 \, , \quad T_-\bar{d} = -\bar{u} \, . \tag{6.21}$$

## 6.5 Building mesons with 2 flavors

Mesons ($q\bar{q}$) can be obtained from diquarks ($qq$) through $u \to -\bar{d}, d \to \bar{u}$ in the second position Alternatively, we can assemble them from an iso-$\frac{1}{2}$ quark and an iso-$\frac{1}{2}$ antiquark state. This is depicted in the book in Fig. 9.8, and the solution is depicted in Fig. 9.9.

Combining the upper part of $q$ (i.e. an $u$) with the upper part of $\bar{q}$ (i.e. an $-\bar{d}$) we have

$$\begin{aligned}
\phi(1,+1) &= -u\bar{d} \\
\phi(1,\ 0\ ) &= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \\
\phi(1,-1) &= +d\bar{u}
\end{aligned} \tag{6.22}$$

as triplet state and (via orthogonality)

$$\phi(0,0) \;=\; \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \tag{6.23}$$

as singlet state. Check that, within the triplet, $T_\pm$ connects $\phi(1,-1)$ with $\phi(1,0)$, and a second application connects $\phi(1,0)$ with $\phi(1,+1)$. Again there are two checks for the singlet state:

$$T_+\frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d}) = \frac{1}{\sqrt{2}}\Big(T_+\{u\}\bar{u} + uT_+\{\bar{u}\} + T_+\{d\}\bar{d} + dT_+\{\bar{d}\}\Big) = \frac{1}{\sqrt{2}}(0 - u\bar{d} + u\bar{d} + 0) = 0$$

$$T_-\frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d}) = \frac{1}{\sqrt{2}}\Big(T_-\{u\}\bar{u} + uT_-\{\bar{u}\} + T_-\{d\}\bar{d} + dT_-\{\bar{d}\}\Big) = \frac{1}{\sqrt{2}}(d\bar{u} + 0 + 0 - d\bar{u}) = 0$$

Mathematicians would summarize our activity by stating $\frac{1}{2} \otimes \bar{\frac{1}{2}} = 1 \oplus 0$ in the spin-notation or by $2 \otimes \bar{2} = 3 \oplus 1$ in the plet-notation.

## 6.6 SU(3) group and algebra

$SU(3)$ is defined as the group of special unitary matrices $U$ with $UU^\dagger = I_3$ and $\det(U) = 1$. A matrix $M \in \mathbb{C}^{3\times3}$ has 18 (real) parameters, hence $SU(3)$ has $18 - 9 - 1 = 8$ parameters.

Similar to what was discussed for $U(2)$ and $SU(2)$, we start from a parameterization of $SU(3)$, and include an extra factor $\exp(\mathrm{i}\alpha_0) = \exp(\mathrm{i}\alpha_0 I_3)$ for $U(3)$. Here the first exp(.) is the standard exponential function, the second one is the matrix exponential function.

For the $SU(3)$ part we need 8 generators [for $N_f$ flavors the group is $SU(N_f)$ which has ${N_f}^2 - 1$ generators]. The generators $T_i$ (with $i = 1...3$ in our case) must be hermitean (to make $U$ unitary) and traceless (to enforce unit determinant). Hence

$$U = \exp(\mathrm{i}\alpha \cdot T) \qquad \text{with} \qquad T_i = T_i^\dagger \;\text{ and }\; \alpha\cdot T \equiv \sum_{i=1}^{8} \alpha_i T_i \tag{6.24}$$

is the standard parameterization for $SU(3)$, if the generators $T_i$ are properly defined. We use

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \quad \lambda_8 = \tfrac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad T_i \equiv \frac{1}{2}\lambda_i \tag{6.25}$$

where the underlying construction principle is as follows. The matrices $\lambda_{1,2,3}$ are just $\sigma_{1,2,3}$ embedded in $\mathbb{C}^{3\times3}$. Next, the pattern of $\sigma_{1,2}$ is repeated in the 13 plane and in the 23 plane. We cannot repeat $\sigma_3$ twice, since it is diagonal and traceless, but the manifold of $3\times3$ diagonal and traceless matrices has only 2 dimensions. Accordingly, we can only add one more matrix

$\lambda_8$ which is diagonal and traceless and "orthogonal" to $\lambda_3$. The normalization in play ensures that every $\lambda_i^2$ has the same trace. Finally, $T_i \equiv \frac{1}{2}\lambda_i$ is analogous to $T_i \equiv \frac{1}{2}\sigma_i$ for $SU(2)$.

Physicswise, $T_3$ acts as $I_3$ on three light (approximately massless) quarks,

$$T_3 u = T_3 \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}+\frac{1}{2}\\0\\0\end{pmatrix} = +\frac{1}{2}u, \quad T_3 d = T_3 \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-\frac{1}{2}\\0\end{pmatrix} = -\frac{1}{2}d, \quad T_3 s = T_3 \begin{pmatrix}0\\0\\1\end{pmatrix} = 0$$

and $T_\pm \equiv T_1 \pm \mathrm{i}T_2 = \frac{1}{2}\lambda_1 \pm \frac{\mathrm{i}}{2}\lambda_2$ raises/lowers $i_3$ as before. The effect of $SU(2)$ on $u \leftrightarrow s$ would be implemented by $\lambda_4, \lambda_5$ and $\lambda' \equiv \mathrm{diag}(1,0,-1)$. The effect of $SU(2)$ on $d \leftrightarrow s$ would be implemented by $\lambda_6, \lambda_7$ and $\lambda'' \equiv \mathrm{diag}(0,1,-1)$. As we learned before, the dimension of the tangent manifold (the "algebra") of $SU(3)$ is 8; hence we cannot have 9 generators. This is why we chose $\lambda_8 \equiv \frac{1}{\sqrt{3}}(\lambda' + \lambda'') = \frac{1}{\sqrt{3}}\mathrm{diag}(1,1,-2)$ to treat $u \leftrightarrow d$ symmetrically.

As usual, we want to determine the maximal commuting set of operators. One finds

$$T^2 \equiv \sum_{i=1}^{8} T_i^2 = \frac{1}{4}\sum_{i=1}^{8}\lambda_i^2 = ... = \frac{4}{3}I_3 \tag{6.26}$$

and this means that every $T_i$ commutes with $T^2$. Unfortunately $[\lambda_i, \lambda_j] \neq 0$ for $i \neq j$, except for $[\lambda_3, \lambda_8] = 0$ (the diagonal ones commute). We choose

$$\Big\{T^2,\ T_3 \equiv \frac{1}{2}\lambda_3,\ Y \equiv \frac{1}{\sqrt{3}}\lambda_8 = \frac{1}{3}\mathrm{diag}(1,1,-2)\Big\} \tag{6.27}$$

as maximal set, where $Y$ is called *hypercharge*. It assigns $u, d, s$ the values $\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}$, respectively. In GETA you learned the GellMann-Nishijima rule $q = i_3 + \frac{1}{2}y$. With $q = \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}$ and $i_3 = \frac{1}{2}, -\frac{1}{2}, 0$ we get $\frac{1}{2}Y = \frac{1}{6}, \frac{1}{6}, -\frac{1}{3}$ for $u, d, s$, respectively, and this matches our assignment.

We are now ready to consider $i_3 y$-diagrams, see Fig. 9.10 in the book. The left panel shows the position of $u, d, s$ (the triplet 3); the right panel shows $\bar{u}, \bar{d}, \bar{s}$ (the antitriplet $\bar{3}$). Note the geometric pattern with $120^\circ = 2\pi/3$ angles at the origin. Later we will "superimpose" such triplets/antitriples onto each other in order to build baryons and mesons. Throughout, $T^2$ defines the "plet", while $T_3, Y$ define the position of the state/particle within the plet. In terms of the number operators $N_u, N_d, N_s$ of quarks, $T_3 = \frac{1}{2}(N_u - N_d)$ and $Y = \frac{1}{3}(N_u + N_d - 2N_s)$.

Now we can define 6 ladder operators, see Fig. 9.12 in the book. Mathematically

$$T_\pm = \frac{1}{2}(\lambda_1 \pm \lambda_2),\ V_\pm = \frac{1}{2}(\lambda_4 \pm \lambda_5),\ U_\pm = \frac{1}{2}(\lambda_6 \pm \lambda_7) \tag{6.28}$$

and the effect on quarks is $T_+ d = u, T_- u = d, V_+ s = u, V_- u = s, U_+ s = d, U_- d = s$, with the other ones annihilating the state (e.g. $T_\pm s = 0, V_\pm d = 0, U_\pm u = 0$). Of course, we want to apply these ladder operators on antiquarks, too. While $SU(3)$ is not pseudoreal, the pseudoreality-trick works for each one of $T, V, U$, separately. Hence $T_+\bar{u} = -\bar{d}, T_-\bar{d} = -\bar{u}$ and $V_+\bar{u} = -\bar{s}, V_-\bar{s} = -\bar{u}$ and $U_+\bar{d} = -\bar{s}, U_-\bar{s} = -\bar{d}$, while the other ones annihilate the state.

## 6.7 Building mesons with 3 flavors

Building mesons with flavor group $SU(3)$ is similar to what we did for $SU(2)$. We superimpose an anti-triplet onto a triplet in exactly the same manner as we superimposed an anti-doublet onto a doublet. A pictorial representation is found in Fig. 9.11 of the book. It is clear that one gets an outer hexagon (with 6 states) plus 3 states in the middle. A closer look will reveal that 2 out of these join the 6 outer states to form an octet, and one guy will be orthogonal.

The trick to disentangle the mess in the middle is – once again – to start at any of the extremes. Now there are 6 such states, and depending on which one we choose, there is one element of $\{T_\pm, U_\pm, V_\pm\}$ that would bring us to the center. This is illustrated in Fig. 6.12 of the book, and in mathematical terms the statement is phrased as

$$\begin{aligned} T_+|d\bar{u}\rangle &= |u\bar{u}\rangle - |d\bar{d}\rangle \;, \qquad & T_-|u\bar{d}\rangle &= |d\bar{d}\rangle - |u\bar{u}\rangle \\ V_+|s\bar{u}\rangle &= |u\bar{u}\rangle - |s\bar{s}\rangle \;, \qquad & V_-|u\bar{s}\rangle &= |s\bar{s}\rangle - |u\bar{u}\rangle \\ U_+|s\bar{d}\rangle &= |d\bar{d}\rangle - |s\bar{s}\rangle \;, \qquad & U_-|d\bar{s}\rangle &= |s\bar{s}\rangle - |d\bar{d}\rangle \;. \end{aligned} \tag{6.29}$$

In the left block the three lines are not independent, e.g. $|u\bar{u}\rangle - |d\bar{d}\rangle - |u\bar{u}\rangle + |s\bar{s}\rangle = -|d\bar{d}\rangle + |s\bar{s}\rangle$. Similarly, in the right block $|d\bar{d}\rangle - |u\bar{u}\rangle - |s\bar{s}\rangle + |u\bar{u}\rangle = -|s\bar{s}\rangle + |d\bar{d}\rangle$. Hence, only 2 out of the 3 states in the middle can be reached from the hexagon, and 1 is orthogonal. The former states form an *octet*, the one orthogonal state is a *singlet*. The two octet members with $i_3 = y = 0$ are not unique; it could be $u\bar{u} - d\bar{d}$ and $u\bar{u} + d\bar{d} - 2s\bar{s}$ [there is a $U(1)$ ambiguity]. The singlet state is unique; its flavor wavefunction is $\phi_{\rm sgl} = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$. Please verify $T_\pm\phi_{\rm sgl} = 0, U_\pm\phi_{\rm sgl} = 0, V_\pm\phi_{\rm sgl} = 0$. This result is depicted in Fig. 9.13 of the book. Hence, the overall result is $3 \otimes \bar{3} = 8 \oplus 1$ in plet-notation (here there is no convenient spin-notation).

Interestingly, the ambiguity of the octet states with zero isospin and zero strangeness is relevant in nature. There are two light meson octets, the $J^P = 0^-$ "pseudoscalar mesons" $(\pi^{-,0,+}, K^0, K^+, K^-, \bar{K}^0, \eta)$, and the $J^P = 1^-$ "vector mesons" $(\rho^{-,0,+}, K^{*0}, K^{*+}, K^{*-}, \bar{K}^{*0}, \omega)$. These states are depicted in Fig. 9.14 of the book. In the former case the flavor content

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \;, \quad \eta \simeq \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \;, \quad \eta' \simeq \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \tag{6.30}$$

matches our calculation in very good approximation (there is only a tiny amount of $\eta - \eta'$ mixing). In the latter case things are significantly different

$$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \;, \quad \omega \simeq \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \;, \quad \phi \simeq s\bar{s} \tag{6.31}$$

since there is strong $\rho^0 - \omega$ mixing, and $\phi$ is almost a pure $s\bar{s}$ state. Recall from GETA that only states with identical quantum numbers can mix.

Such information on the flavor content is useful to make predictions, e.g. for the meson masses. Suppose the quarks interact with each other via a spin-spin coupling, that is

$$U \propto \frac{g}{m_i}\vec{S}_i \cdot \frac{g}{m_j}\vec{S}_j \propto \frac{\alpha_{\rm st}}{m_i m_j}\vec{S}_i \cdot \vec{S}_j \tag{6.32}$$

would be a contribution to the total energy ($g$ is the strong coupling that replaces $e$, and $\alpha_{\rm st}$ replaces $\alpha_{\rm em}$). For heavy quarks it is reasonable to describe their interaction through a potential, for light quarks it is not. Ignoring this for a moment, we would expect the meson mass $M$ to relate to the masses $m_i, m_j$ of the flavors $i, j$ it is made from via

$$M(q_i\bar{q}_j) = f(m_i) + f(m_j) + \frac{\rm const}{m_i m_j}\langle\vec{S}_i \cdot \vec{S}_j\rangle \tag{6.33}$$

where most physicists would replace the function $f(.)$ by the identity. What is $\langle\vec{S}_i\cdot\vec{S}_j\rangle$ ? The trick is to write $\langle\vec{S}_i\cdot\vec{S}_j\rangle=\frac{1}{2}[\langle\vec{S}^2\rangle-\langle\vec{S}_i^2\rangle-\langle\vec{S}_j^2\rangle]$ where $\vec{S}$ is the total spin of the $q\bar{q}$ system (i.e. of the meson). In this form each term is known: $\langle\vec{S}_i^2\rangle=s_i(s_i+1)=\frac{3}{4}$ for each quark $q_i$ (and ditto each antiquark $\bar{q}_j$), while $\langle\vec{S}^2\rangle=s(s+1)$ is 0 for pseudoscalar mesons and 2 for vector mesons. Overall, $\langle\vec{S}_i\cdot\vec{S}_j\rangle$ is found to be $\frac{1}{2}[0-\frac{3}{2}]=-\frac{3}{4}$ for pseudoscalar mesons and $\frac{1}{2}[2-\frac{3}{2}]=\frac{1}{4}$ for vector mesons. Hence we reach the "predictions" for $q_i\bar{q}_j$ mesons

$$M_P=m_i+m_j-\frac{3}{4}\frac{A}{m_im_j}\ ,\quad M_V=m_i+m_j+\frac{1}{4}\frac{A}{m_im_j} \tag{6.34}$$

and – even if we do not trust the first two terms – we have the more robust prediction

$$M_V-M_P=\frac{A}{m_im_j}>0 \tag{6.35}$$

which relates the *hyperfine splitting* to the product of the quark masses.

The problem with the first form is that fitting to experimental data suggests so-called "constituent quark masses" $m_{u,d}\sim 300\,\mathrm{MeV}$, $m_s\sim 500\,\mathrm{MeV}$ which have nothing to do with the "current quark masses" listed in Sec. 2. The latter quark masses are fundamental quantities, while e.g. the $u$ constituent quark mass depends on the meson or baryon the $u$ sits in.

## 6.8 Building diquarks and baryons with 3 flavors

We are now ready to combine two quarks with $SU(3)$ flavor group rather than $SU(2)$. Superimposing the center of the second down-triangle onto the edges of the first down-triangle gives the structure shown in Fig. 9.15 of the book. In other words, the figure tells us that $3\otimes3=6\oplus\bar{3}$ in plet-notation (again, there is no spin-notation for $qq$). The 6 is symmetric under flavor exchange, the $\bar{3}$ is antisymmetric (note the difference between 3 and $\bar{3}$, one is a down-triangle, one is an up-triangle). The flavor content is determined in the usual fashion. One starts at the edges of the 6, i.e. the states $uu$, $dd$, $ss$ are readily identified. Next, one uses $T_\pm,U_\pm,V_\pm$ to reach the remaining three positions of the 6. This gives $\frac{1}{\sqrt{2}}(ud+du)$, $\frac{1}{\sqrt{2}}(ds+sd)$, $\frac{1}{\sqrt{2}}(su+us)$, respectively. The three positions on $\bar{3}$ (which are at identical $i_3y$-coordinate positions) are $\frac{1}{\sqrt{2}}(ud-du)$, $\frac{1}{\sqrt{2}}(ds-sd)$, $\frac{1}{\sqrt{2}}(su-us)$, for reasons of orthogonality.

With these results in hand, we can now reduce out baryons in flavor space. In other words, $3\otimes3\otimes3=(6\oplus\bar{3})\otimes3$, and our task is to work out irreducible representations ("irreps") of $6\otimes3$ and $\bar{3}\otimes3$. The three steps of this procedure are illustrated in Fig. 9.16 of the book. Panel (a) just illustrates the break-up into two sub-tasks. Panel (b) depicts the process of reducing $6\otimes3=10+8$. Here we are called to overlay the center of the 3 onto each node of the 6; the result is shown in the middle (six points are two-fold degenerate, one three-fold). The flavor assignments in the nodes of the 10 follow from the edges by using the ladder operators (one is shown in the Fig.). The flavor assignments in the nodes of the 8 follow from orthogonality (with an ambiguity in the middle; one node is shown in the Fig.). Panel (c) depicts the process of reducing $\bar{3}\otimes3=8+1$. Again, we start by overlaying the center of the 3 onto each node of the $\bar{3}$. The result is nine states with a hexagonal boundary and a three-fold occupied center (not shown in the Fig.). They are disentangled into an octet and a singlet, as stated above.

Note that the two 8 emerging in this reduction are *not identical*. This follows from a look at the symmetry properties of these (multi)plets. The 8 that emerged from $6\otimes3=10+8$ is mixed symmetric (there is no overall symmetry, but it is symmetric under exchange of the first two flavors). The 8 that emerged from $\bar{3}\otimes 3 = 8+1$ is mixed anti-symmetric (there is no overall symmetry, but it is anti-symmetric under exchange of the first two flavors). In formulas

$$3\otimes 3\otimes 3 = \underbrace{10}_{\text{tot.symm}} \oplus \underbrace{8}_{\text{mix.symm}} \oplus \underbrace{8}_{\text{mix.anti}} \oplus \underbrace{1}_{\text{tot.anti}} \tag{6.36}$$

and the requirement of total anti-symmetry determines the flavor wavefunction of the $1_{\text{tot.anti}}$ uniquely as $\frac{1}{\sqrt{6}}(uds-usd+dsu-dus+sud-sdu)$. Please check yourself that this combination vanishes if any of the operators $T_\pm, V_\pm, U_\pm$ is applied.

Finally, recall that all of this concerns the factor $\phi_{\text{flavor}}$ in (6.15). Obviously, if $\phi_{\text{flavor}}$ has a mixed symmetry pattern under exchange of two quarks, the remaining three factors (together) must have an appropriately matched (mixed) symmetry pattern to ensure that $\psi$ is totally anti-symmetric under any (complete) exchange of two quarks. In Sec. 7 we will learn that $\xi_{\text{color}}$ is totally anti-symmetric by itself. In the ground state ($\ell = 0$) the factor $\eta_{\text{space}}$ is totally symmetric by itself. The product $\phi_{\text{flavor}}\cdot\chi_{\text{spin}}$ in the baryon wavefunction is totally symmetric under exchange of any two quarks. Recall that $\chi_{\text{spin}}$ was $\chi_{3/2,\text{tot.symm}} \doteq \chi_{4,\text{tot.symm}}$, as well as $\chi_{1/2,\text{mix.symm}} \doteq \chi_{2,\text{mix.symm}}$ and $\chi_{1/2,\text{mix.anti}} \doteq \chi_{2,\text{mix.anti}}$. Overall, we find the possibilities

$$\psi = \underbrace{\phi_{10}\cdot\chi_4}_{J^P=\frac{3}{2}^+\text{ baryon decuplet}} \qquad \text{or} \qquad \psi = \underbrace{\frac{1}{\sqrt{2}}\Big(\phi_{8,S}\chi_{2,S}+\phi_{8,A}\chi_{2,A}\Big)}_{J^P=\frac{1}{2}^+\text{ baryon octet}} \tag{6.37}$$

where $S$ stands for "mixed symm", $A$ stands for "mixed anti", and the missing subtext in $\phi_{10}\cdot\chi_4$ is supposed to indicate that each factor is totally symmetric. The "baryon decuplet" comprises the states $\Delta, \Sigma^*, \Xi^*, \Omega$, while the "baryon octet" contains the states $N, \Sigma\,\&\,\Lambda, \Xi$. These states are depicted in Fig. 9.17 of the book. Note that "p (uud)" just gives the net flavor content of the proton; the "(.)" stands for "an undisclosed symmetry pattern" of the argument.

It goes without saying that also for $SU(3)$ as symmetry group knowing the factors $\phi_{\text{flavor}}$ and $\chi_{\text{spin}}$ in (6.15) allows for simple predictions for masses and magnetic moments of baryons built from $u, d, s$ quarks. Historically, this was important to convince the particle physics community of the presence of the factor $\xi_{\text{color}}$ and thus of the concept of color charges.

## 6.9 Summary

- Flavor symmetry is a symmetry among 2 or 3 approximately massless quarks ($m_q \ll 1\,\text{GeV}$).
- The group is $SU(2)$ or $SU(3)$, respectively, and it is global (no space-time dependence).
- The number of parameters is 3 and 8, respectively, in any valid parameterization.
- A convenient parameterization is exp(.), with 3 or 8 hermitean and traceless generators $T_i$.
- Meson flavor states are built from $2\otimes\bar{2}$ or $3\otimes\bar{3}$; they correspond to bound states in nature.
- Diquark flavor states are built from $2\otimes 2$ or $3\otimes 3$; they are an intermediate step for baryons.
- Baryon flavor states are built from attaching "$\otimes 3$" to each diquark irrep, and reducing them.
- From $\phi_{\text{flavor}}$ and $\chi_{\text{spin}}$ in (6.15) simple predictions for masses and magnetic moments follow.

# 7 Strong interactions via local SU(3) gauge group

## 7.1 Local gauge invariance in QED and QCD

In classical electrodynamics (TP 2) you learned that the theory is invariant under a change

$$\phi(x) \longrightarrow \phi'(x) \equiv \phi(x) - \partial_t \chi(x) \qquad \text{and} \qquad \vec{A}(x) \longrightarrow \vec{A}'(x) \equiv \vec{A}(x) + \vec{\nabla}\chi(x) \tag{7.1}$$

of the scalar and vector potentials, with $x = x^\bullet = (t, \vec{x})$. Using the four-vector potential $A_\bullet = (\phi, -\vec{A})$ and the four-derivative $\partial_\bullet = (\partial_0, \vec{\nabla})$, we may rewrite this as

$$A_\mu(x) \longrightarrow A'_\mu(x) \equiv A_\mu(x) - (\partial_\mu \chi)(x) \tag{7.2}$$

where no change of the argument is implied (i.e. $x' = x$).

In quantum mechanics (TP 3) you learned that the gauge invariance of electrodynamics manifests itself in a local phase transformation of the wavefunction

$$\psi(x) \longrightarrow \psi'(x) \equiv U(x)\psi(x) \qquad \text{with} \qquad U(x) \equiv \exp(\mathrm{i}q\chi(x)) \ \in U(1) \tag{7.3}$$

where $q$ is the charge of the particle. The possible complex phases form the one-dimensional Lie group $U(1)$ with a single generator (the identity in tangent space). This transformation is *local* because $U = U(x)$, while the space-time independent $U$ discussed in Sec. 6 were *global.*

Suppose we have the funny idea of requesting the Dirac equation to be invariant under the local gauge transformation (7.3). Plugging $\psi'(x) = \exp(\mathrm{i}q\chi(x))\psi(x)$ into $\mathrm{i}\gamma^\mu\partial_\mu\psi'(x) = m\psi'(x)$ yields $\mathrm{i}\gamma^\mu\partial_\mu\{\exp(\mathrm{i}q\chi(x))\psi(x)\} = m\exp(\mathrm{i}q\chi(x))\psi(x)$. Applying the product rule, and left-multiplying the result with $\exp(-\mathrm{i}q\chi(x))$ eventually leads to

$$\mathrm{i}\gamma^\mu\Big[\partial_\mu + \mathrm{i}q(\partial_\mu\chi)(x)\Big]\psi(x) - m\psi(x) = 0 \tag{7.4}$$

and this equation differs from the Dirac equation for $\psi(x)$ by an extra term $-\gamma^\mu q(\partial_\mu\chi)(x)\psi(x)$ on the left-hand side. This undesired term looks deceptively like the last term in (7.2) times $\psi(x)$. What could we do ? We could stipulate that the original Dirac equation is actually

$$\mathrm{i}\gamma^\mu\Big[\partial_\mu + \mathrm{i}qA_\mu(x)\Big]\psi(x) - m\psi(x) = 0 \tag{7.5}$$

and try the combined effect of applying both the gauge transformation (7.3) for the wavefunction and (7.2) for the vector potential. Plugging these two transformations into (7.5) with the replacement $\psi \to \psi'$ and $A_\mu \to A'_\mu$, one notices two unwanted terms. Fortunately they have opposite sign, so they cancel (please verify this statement). We have seen (7.5) before; in (3.40) it was introduced under the headings of "minimal substitution". Hence, we learn that local gauge invariance is the principle behind minimal substitution.

In electrodynamics the question arises whether there is any physical entity associated with the abelian gauge potential $A_\mu$ or whether it is mere mathematical trickery. The electromagnetic field $F^{\mu\nu}$ is found to transport energy and momentum, but attributing a physical reality to $A_\mu$ is hindered by its gauge dependence (we use Feynman gauge). After the electromagnetic potential is quantized, "photons" are the excitations of $A_\mu$ (see QFT course).

Pauli was the first physicist to consider a generalization of (7.3) to $U(x) \in SU(2)$ or $U(x) \in SU(3)$ for each $x \in \mathbb{R}^4$, i.e. to non-Abelian gauge groups. Hence (7.3) is replaced by

$$\psi(x) \longrightarrow \psi'(x) \equiv U(x)\psi(x) \qquad \text{with} \qquad U(x) \equiv \exp(\mathrm{i}g\chi(x)) \ \in SU(N_c) \tag{7.6}$$

where $\chi(x)$ is a traceless hermitean $N_c \times N_c$ matrix (the *Lie algebra element*), defined by

$$\chi(x) = \vec{\chi}(x) \cdot \vec{T} = \sum_{a=1}^{N_c^2-1} \chi^a(x) T^a \ . \tag{7.7}$$

Throughout, $N_c$ is the number of "color" charges, i.e. $N_c = 2$ for $SU(2)$, or $N_c = 3$ for $SU(3)$. The dot-product in this equation has nothing to do with three-dimensional space; it involves 3 terms for $SU(2)$, and 8 terms for $SU(3)$. The $T^a$ are the generators (i.e. hermitean traceless $N_c \times N_c$ matrices) known from Sec. 6. Specifically, we have $T^a = \sigma^a/2$ with the Pauli matrices $\sigma^a$ and $a = 1...3$ for $SU(2)$, and we have $T^a = \lambda^a/2$ with the GellMann matrices $\lambda^a$ and $a = 1...8$ for $SU(3)$. In (7.6) the formerly-electric "charge" $q$ is replaced by a generic non-Abelian "coupling" $g$. In weak interactions it will be specified as $g \to g_{\mathrm{wk}}$ or $g_W$, in strong interactions it will be specified as $g \to g_{\mathrm{st}}$ or $g_C$. The Dirac equation takes the form

$$\mathrm{i}\gamma^\mu \Big[\partial_\mu + \mathrm{i}gA_\mu(x)\Big]\psi(x) - m\psi(x) = 0 \qquad \text{with} \qquad A_\mu(x) = \sum_a A_\mu^a(x) T^a \tag{7.8}$$

where the local non-Abelian gauge potential $A_\mu(x)$ is a (space-time dependent) superposition of the 3 or 8 generators $T^a$ (which are the same everywhere). The transformation law

$$A_\mu^a \longrightarrow A_\mu^{a\prime} \equiv A_\mu^a - \partial_\mu \chi^a - g f^{abc} \chi^b A_\mu^c \tag{7.9}$$

of the gauge potential (with $b, c$ summed over) involves the structure constant $f^{abc}$ of the gauge group; they are defined via the commutation relation $[T^a, T^b] = \mathrm{i} f^{abc} T^c$.

The interaction term in the Dirac equation (7.8) reveals the quark-quark-gluon vertex as

$$g\gamma^\mu A_\mu(x)\psi(x) = g\gamma^\mu A_\mu^a(x) \frac{\lambda^a}{2} \psi(x) \tag{7.10}$$

with implicit summation over the "adjoint index" $a$. In consequence, we find the recipe

$$\boxed{-\mathrm{i}q\gamma^\kappa \qquad \longrightarrow \qquad -\mathrm{i}g\gamma^\kappa \frac{\lambda_{ij}^a}{2}} \tag{7.11}$$

for upgrading the QED Feynman rule (fermion-fermion-photon vertex) to the QCD Feynman rule (quark-quark-gluon vertex). Here, $a \in \{1, ..., N_c^2 - 1\}$ is the "adjoint index" of the gluon, while $i, j \in \{1, ..., N_c\}$ are the "fundamental indices" of the outgoing/incoming quark. In QCD the $3 \times 3$ matrix $\lambda^a$ is to be evaluated at position $ij$, with $a \in \{1...8\}$ and $i, j \in \{1...3\}$.

A peculiarity of non-Abelian groups is that the *same* gauge coupling $g$ (as defined through the quark-quark-gluon vertex) also appears in the Feynman rules for 3-gluon and 4-gluon vertices (to which there is no counterpart in QED), see Fig. 10.1 of the book. There are Feynman rules for these 3-gluon and 4-gluon vertices, too, and there is a Feynman rule for the gluon propagator. For the photon propagator we had $-\mathrm{i}\eta_{\mu\nu}/q^2$, for the gluon propagator it will be $-\mathrm{i}\eta_{\mu\nu}\delta^{ab}/q^2$. This means it cannot change its adjoint index ($a, b \in \{1...8\}$ in QCD) in flight.

## 7.2 Some key concepts of QCD

The bottom line of the previous subsection is that “color” is a new *internal degree of freedom*, with 3 new charges. For the lack of a better idea, we call them “red”, “green”, “blue” (r, g, b). And an anti-quark can have the colors “anti-red”, “anti-green”, “anti-blue” ($\bar{\mathrm{r}}, \bar{\mathrm{g}}, \bar{\mathrm{b}}$). So

$$r \doteq \begin{pmatrix}1\\0\\0\end{pmatrix} \;, \quad g \doteq \begin{pmatrix}0\\1\\0\end{pmatrix} \;, \quad b \doteq \begin{pmatrix}0\\0\\1\end{pmatrix} \;, \tag{7.12}$$

spans an internal space, and quarks transform in this 3-dimensional “fundamental” representation of $SU(3)$. This has consequences for the available color structures of a $q\bar{q}$ combination (or $q_k\bar{q}_\ell$ combination, with flavor indices $k, \ell$). You recall that the three flavors of $q$ were presented as a 3 (down-triangle) in the $i_3y$-plane. In complete analogy, the three colors of a quark can be presented as a 3 (down-triangle) in an analogous plane (some people call it the $i_3^C y^C$-plane). And the three anti-colors of an anti-quark can be presented as a $\bar{3}$ (up-triangle) in this artificial plane. See Fig. 10.2 of the book for an illustration.

Fig. 10.7 of the book shows that the way how this color-anticolor structure is reduced out is an exact match to what we did in Sec. 6 for flavor. The color structure of a $q\bar{q}$ pair decomposes into an octet and a singlet. Mathematically, it is again $3 \otimes \bar{3} = 8 \oplus 1$, exactly as in the previous section, with the replacement $u \to r, d \to g, s \to b$. The eight states in the octet are

$$r\bar{g}\,, \quad g\bar{r}\,, \quad g\bar{b}\,, \quad b\bar{g}\,, \quad b\bar{r}\,, \quad r\bar{b}\,, \quad \frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})\;, \quad \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b}) \tag{7.13}$$

and it is easy to check that the ladder operators $T_\pm, U_\pm, V_\pm$ transform them into each other. The singlet state must be orthogonal to any state of the octet, and indeed

$$\frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b}) \qquad \longleftrightarrow \qquad \xi_{\mathrm{color}}(q\bar{q}) \tag{7.14}$$

is a carbon copy of our previous result (again any of $T_\pm, U_\pm, V_\pm$ annihilates this state). Hence, if a $q\bar{q}$ state is color neutral, its color wavefunction must be of this totally symmetric type.

Gluons behave, as far as color is concerned, a bit like a $q\bar{q}$ pair, though not quite. A gluon may take any state of (7.13), but not (7.14). In other words, it may carry any color-different-anticolor combination [first six entries in (7.13)], or it carries a linear combination of $\mathrm{r}\bar{\mathrm{r}}, \mathrm{g}\bar{\mathrm{g}}, \mathrm{b}\bar{\mathrm{b}}$ which is orthogonal to (7.14) [last two entries]. This is why there are 8 gluons (in general $N_c^2 - 1$ for $N_c$ colors). Mathematicians and theoretical physicists say that gluons transform in the “adjoint representation” (which acts on a $N_c^2 - 1$ dimensional vectorspace). As a by-product you see why the “intuitive understanding” of the adjoint nature of the gluon (in the book depicted in Fig. 10.4) is limited. It yields a good illustration for the “color-different-anticolor” gluons, i.e. the first six entries in (7.13) [in the figure it is $\mathrm{b}\bar{\mathrm{r}}$ or $\mathrm{r}\bar{\mathrm{b}}$]. But it is blind to the details of a gluon in the middle of the octet. For instance, if the quark at the upper vertex stays $r$, and the one at the lower vertex stays $b$, this picture is not sensitive to the difference between a $\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ or $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$ gluon (which works), and a $\frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$ “gluon” (which does not exist). In short, the color-octet (“adjoint”) nature of the gluon is important.

## 7.3 Color confinement in QCD at large distances

In QED the fermion-fermion-photon vertex yields a factor $-\mathrm{i}q\gamma^\kappa$, if the electric charge of the fermion line is $q$. In QCD we plug in a factor $-\mathrm{i}g\gamma^\kappa\frac{\lambda^a_{ij}}{2}$, if the outgoing gluon has color $i$, and the incoming one has color $j$. The adjoint index $a$ is a superficial index, like $\kappa$, i.e. you can attribute it at your discretion, since it will be matched by an identical index of the gluon propagator and summed over. In any case, no momentum shows up in the electron-electron photon vertex (QED) and the quark-quark-gluon vertex (QCD).

Nevertheless, if you define the QED or QCD coupling strength by a potential

$$V_{\mathrm{QED}}(r) = -\frac{e^2}{4\pi r} = -\frac{\alpha_{\mathrm{em}}}{r} \qquad \text{or} \qquad V_{\mathrm{QCD}}(r) = -\frac{g^2}{4\pi r} = -\frac{\alpha_{\mathrm{st}}}{r} \quad [C_F \text{ missing}] \tag{7.15}$$

the coupling $e^2$ or $g^2$ depends a bit on the distance $r$. Since it is common to let the (renormalized) coupling depend on a scale $\mu$ or $\mu^2$ rather than a distance, one might write

$$V_{\mathrm{QED}}(r) = -\frac{\alpha_{\mathrm{em}}(\mu \sim 1/r)}{r} \qquad \text{or} \qquad V_{\mathrm{QCD}}(r) = -\frac{\alpha_{\mathrm{st}}(\mu \sim 1/r)}{r} \quad [C_F \text{ missing}] \tag{7.16}$$

where also the notation with $\alpha_{\mathrm{em}}(\mu^2 \sim 1/r^2)$ or $\alpha_{\mathrm{st}}(\mu^2 \sim 1/r^2)$ is frequently found. The "running" of the coupling $e^2(\mu^2)$ or $g^2(\mu^2)$ comes from loop-diagrams like those in Fig. 10.10 and 10.11 of the book. Their evaluation is a key topic in a course on QFT.

What is of relevance in this course, are two qualitative differences between QED and QCD. In QED this "running" is an extremely mild effect, in QCD it is very pronounced. Moreover, they work in opposite direction in these two theories. In QED the coupling becomes stronger at short distances, in QCD it becomes weaker (and vice versa at large distances).

This is illustrated in Figs. 10.12 and 10.14, respectively. The QED "running" is shown in Fig. 10.12. At large distances, $q^2 \downarrow 0$, and $\alpha_{\mathrm{em}}(q^2 \downarrow 0)$ tends to about $1/137.036 \simeq 0.0073$ at the left-infinite boundary (dotted line). At the highest available energies, $q^2 \simeq (1\,\mathrm{TeV})^2$, it increases to about 0.008 (at the $Z$ mass it is about $1/127 \simeq 0.0075$). The QCD "running" is shown in Fig. 10.14, and it is far more pronounced (compare the range of the ordinates). At large distances, $\alpha_{\mathrm{st}}(q^2 \downarrow 0)$ seems to diverge at the left-infinite boundary [that's not true, it plateaus at $q^2 \simeq (200\,\mathrm{MeV})^2 \simeq (1\,\mathrm{fm})^{-2}$]. At short distances, in the figure up to $q^2 \simeq (1\,\mathrm{TeV})^2$, the coupling seems to decrease monotonically (that's true for QCD).

The reason for this difference is the gluon self-interaction which, in turn, is due to it carrying a color charge degree of freedom (in the adjoint representation) to which it couples. In nature, only color neutral objects are observed, e.g. three quarks can be in the singlet state

$$\frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr) \qquad \longleftrightarrow \qquad \xi_{\mathrm{color}}(qqq) \tag{7.17}$$

which, once more, is a carbon-copy of our result from Sec. 6. The interesting news is that several gluons may jointly form a color-singlet state, e.g. the first six elements of (7.13) can form such a state. Hence, if dynamics allows for this, gluons may form so-called glue-balls, i.e. color-neutral bound states that might appear in asymptotic (i.e. "in" or "out") states.

Equation (7.16) holds for $r \ll 1\,\mathrm{fm}$, i.e. in a regime where $\alpha_{\mathrm{st}} \ll 1$ [modulo a so-far undiscussed factor 4/3]. For $r > 0.1\,\mathrm{fm}$ another term kicks in, and the potential changes to

$$V_{\mathrm{QCD}}(r) \simeq -C_F\frac{\alpha_{\mathrm{st}}(\mu \sim 1/r)}{r} + \sigma r \qquad \text{with} \qquad C_F \equiv \frac{4}{3} \tag{7.18}$$

as shown in Fig. 10.26 of the book [full line, the first term is the dotted line]. The constant $\sigma \sim 1\,\mathrm{GeV/\,fm}$ is called *string-tension*. It generates an additional attractive force which is independent of $r$, and surprisingly large (about 100'000 N in SI units). This qualitative change happens just when $\alpha_{\rm st}$ is about to become *really strong*, i.e. at distances $r \sim 1\,\mathrm{fm}$.

There is yet another effect in full QCD, known as "string-breaking", see Fig. 10.9 in the book. If we separate a $q\bar{q}$ pair, initially their interaction is governed by eqn. (7.16). However, as $r$ keeps growing, the second term in eqn. (7.18) becomes important, and the "string" (i.e. the color flux-tube between the $q\bar{q}$ pair) stores an increasing amount of energy. At $r \sim 1\,\mathrm{fm}$ it eventually becomes energetically more favorable to form an additional $q\bar{q}$ pair which allows for breaking the string into two shorter strings. This mechanism is believed to be at the heart of color confinement, i.e. only color-neutral states can show up as asymptotic states.

## 7.4 Running couplings and the definitions of $\alpha_{\rm em}$ and $\alpha_{\rm st}$

The process of *renormalization* is easiest to explain in QED (and even there it involves concepts which are beyond the level of this course). Let us take another look at Fig. 10.11. If we define the coupling strength $e$ or $e^2$ or $\alpha_{\rm em}$ through the the scattering of two electrons, then the result depends (very mildly) on the Mandelstam variable $t = q^2$. This is what experiment finds, see (again) Fig. 10.12. On the theory side we can calculate Feynman diagrams, e.g. those in Fig. 10.10 or 10.11. These involve the "bare" positron charge $e_0$, so far denoted by $e$. While $e_0$ is $q$-independent, the resulting coupling runs, i.e. $e = e(q^2)$, as long as the overall form is $e(q^2)/q^2$, which it is. In Fig. 10.11 all the "bubbles" are *resummed* into a hatched "blob" which, in turn, amounts to a renormalized coupling. The key idea is that we should not identify $e_0^2/(4\pi)$ with the measured $\alpha_{\rm em} \simeq 1/137.036$, but rather $e^2(q^2 \downarrow 0)/(4\pi)$ should be identified with this number. And theory should, of course, work out the relationship $e_0^2 \longleftrightarrow e^2(q^2)$.

In the book, the idea for this resummation is presented. The tree-level diagram is $P_0 = e_0^2/q^2$. The one-loop diagram is $P_0\pi(q^2)P_0$, where $\pi(q^2)$ is the one-loop bubble. The two-loop diagram is $P_0\pi(q^2)P_0\pi(q^2)P_0$, and so on. Hence, this specific class of diagrams can be summed

$$\begin{aligned} P &\equiv P_0 + P_0\pi(q^2)P_0 + P_0\pi(q^2)P_0\pi(q^2)P_0 + \ldots \\ &= P_0\left[1 + \pi(q^2)P_0 + \pi(q^2)P_0\pi(q^2)P_0 + \ldots\right] = P_0\frac{1}{1-\pi(q^2)P_0} \end{aligned} \tag{7.19}$$

since the rules of the geometric series apply. Our goal was to identify $P = e^2(q^2)/q^2$, hence

$$e^2(q^2) = \frac{e_0^2}{1 - e_0^2\pi(q^2)/q^2} = \frac{e_0^2}{1 - e_0^2\Pi(q^2)} \tag{7.20}$$

with $\Pi(q^2) = \pi(q^2)/q^2$. This relation can be inverted to read (note the sign in the denominator)

$$e_0^2 = \frac{e^2(q^2)}{1 + e^2(q^2)\pi(q^2)/q^2} = \frac{e^2(q^2)}{1 + e^2(q^2)\Pi(q^2)} \tag{7.21}$$

up to terms $O(e^4)$. From this we learn that the full right-hand side does not depend on $q^2$, and a few mathematical operations thus yield [as always up to $O(e^4)$ terms] the relation

$$e^2(q^2) = \frac{e^2(\mu^2)}{1 - e^2(\mu^2)[\Pi(q^2) - \Pi(\mu^2)]} \tag{7.22}$$

which tells us how the fine-structure constant at one $q^2$ differs from its sibling at a reference momentum squared, $\mu^2$. Hence the theory job is to determine the square bracket, and they find $[\Pi(q^2) - \Pi(\mu^2)] \simeq \ln(q^2/\mu^2)/(12\pi^2)$ for QED. Overall, we have

$$\alpha_{\mathrm{em}}(q^2) \simeq \frac{\alpha_{\mathrm{em}}(\mu^2)}{1 - \alpha_{\mathrm{em}}(\mu^2)\ln(q^2/\mu^2)/(3\pi)} \tag{7.23}$$

where we should stress that this relation holds only at the one-loop level. The important thing is the *minus sign* in the denominator (for QED). As we increase $q^2$ the denominator gets smaller, and the QED coupling constant thus larger, as seen in Fig. 10.12.

Now we repeat this idea for QCD. In the book the leading diagrams are shown in Fig. 10.13. The self-energy correction of the photon had one contribution; the self-energy correction of the gluon has three such contributions. The story is the same as before, hence we find (7.22) with $e^2(.) \to g^2(.)$ replaced, except that $[\Pi(q^2) - \Pi(\mu^2)]$ has a different factor ahead of $\ln(q^2/\mu^2)$, not $1/(12\pi^2)$ any more, but $-[11N_c - 2N_f]/(48\pi^2)$. Here $N_c$ is the number of colors ($N_c = 3$ in the SM), and $N_f$ is the number of fermions which couple to gluons ($N_f = 6$ in the SM). Hence

$$\alpha_{\mathrm{st}}(q^2) \simeq \frac{\alpha_{\mathrm{st}}(\mu^2)}{1 + [11N_c - 2N_f]\alpha_{\mathrm{st}}(\mu^2)\ln(q^2/\mu^2)/(12\pi)} \tag{7.24}$$

with a plus sign in the denominator, in contradistinction to (7.23). This means that $\alpha_{\mathrm{st}}(q^2)$ decreases as a function of $q^2$, see again Fig. 10.14 in the book. In particular

$$\alpha_{\mathrm{st}}(q^2 \equiv M_Z^2) = 0.1184 \pm 0.0004 = 0.1184(4) \tag{7.25}$$

has been measured very precisely by various experiments and theory (in lattice QCD). There is some discussion whether this precision is really warranted, the PDG gives 0.1179(10).

## 7.5 Asymptotic freedom in QCD at short distances

One may consider generalizations of QCD, with arbitrary $N_c, N_f$. From (7.24) it follows that the sign of $[11N_c - 2N_f]$ is crucial for the behavior of the theory. For $N_f < 11N_c/2$, or $N_f \leq 16$ for $N_c = 3$, one has *asymptotic freedom*, which is the sound behavior (in the UV). With more flavors one is back to the unhealthy high-energy properties of QED (with a Landau pole).

Physicswise, a small coupling at large $q^2$ is convenient. It means that "partons" are quasi-free particles if studied at high enough energies. In Fig. 10.14 we see $\alpha_{\mathrm{st}}(q^2 \sim (300\,\mathrm{GeV})^2) \simeq 0.1$, which is small but not incredibly small. It doubles to 0.2 at about $q^2 \sim (6\,\mathrm{GeV})^2$, and for even lower $q^2$ it becomes too large to allow for a perturbative treatment.

## 7.6 Summary

- The Feynman rule for the $qqg$ vertex in QCD is similar to the $ff\gamma$ rule in QED, see (7.11).
- The Feynman rule for the $ggg$ vertex in QCD is complicated, since it involves momenta.
- The Feynman rule for the $gggg$ vertex in QCD is complicated, since it involves many $f_{abc}$.
- The Feynman rule for the gluon propagator in Feynman gauge is $-\mathrm{i}\eta_{\mu\nu}\delta^{ab}/q^2$.
- Important qualitative features are asymptotic freedom, running coupling, confinement.
- The NRQCD potential (7.18) explains some heavy-quark physics, but not string breaking.

# 8 Weak interactions via local SU(2) gauge group

## 8.1 Parity conservation for interactions with V structure

Quick summary of QED, QCD, and QFD (quantum flavor dynamics, i.e. weak interactions).

QED: $\bar{u}(e^-,p')(+\mathrm{i}e\gamma^\mu)u(e^-,p)\cdot(-\mathrm{i}\eta_{\mu\bullet}/q^2)$, discr. symmetries $C,P,T$ individ. obeyed

QCD: $\bar{u}(s,j,p')(-\mathrm{i}g_{\mathrm{st}}\gamma^\mu\lambda^a_{ji})u(s,i,p)\cdot(-\mathrm{i}\eta_{\mu\bullet}\delta^{a\bullet}/q^2)$, discr. symmetries $C,P,T$ individ. obeyed

QFD: $\bar{u}(e^-,p')(???)u(\nu_e,p)\cdot(???)$, $C$ nearly max. violated, $P$ max. violated, $T$ nearly obeyed

Physicswise the differences are evident from comparing Figs. 11.1 and 11.3 in the book:
- Photon never changes flavor, neither on the lepton side nor on the quark side.
- $W^\pm$ boson must change flavor ("exactly vertically" on leptons, "any floor change" on quarks).

Since $P$ is *maximally violated* in charged-current weak interactions, it pays to recap the role of parity. Parity is involutionary and unitary, hence it is hermitean and has eigenvalues $\pm1$ only. From (3.51) we know $P(f\bar{f})=-1$, and analogously one can show $P(b\bar{b})=1$. In a QFT course one derives these properties from the spin-statistics theorem (which links $b\leftrightarrow$ integer-spin and $f\leftrightarrow$ half-integer-spin). In the SM it is a generally accepted convention to have $P(e^-)=P(\nu_e)=P(q)=1$ and $P(e^+)=P(\bar{\nu}_e)=P(\bar{q})=-1$. In the QFT course one may show $P(\gamma)=P(g)=P(W^\pm)=P(Z)=-1$.

Next it pays to recap where parity conservation in QED and QCD comes from. Let us again take a look at Fig. 11.1 in the book. The invariant amplitude is $-\mathrm{i}M=\bar{u}(p_3)(+\mathrm{i}e\gamma^\mu)u(p_1)\cdot(-\mathrm{i}\eta_{\mu\nu}/q^2)\cdot\bar{u}(p_4)(-\mathrm{i}q\gamma^\nu)u(p_2)$, where $e$ is positron charge, one $q$ is the charge of quark field, while the other $q^2$ is the squared momentum of the photon propagator. Using the current notation this can be rewritten as $-\mathrm{i}M=-\mathrm{i}eqj^\mu\eta_{\mu\nu}k^\nu/q^2$, with the electron current $j^\mu\equiv\bar{u}(p_3)\gamma^\mu u(p_1)$, and the quark current $k^\nu\equiv\bar{u}(p_4)\gamma^\nu u(p_2)$. The statement is that this structure conserves parity, and things are identical for strong interactions.

Parity acts on a $u$-spinor as $u\mapsto Pu\equiv\gamma^0u$, hence $\bar{u}\mapsto(\gamma^0u)^\dagger\gamma^0=u^\dagger\gamma^{0\dagger}\gamma^0=u^\dagger=\bar{u}\gamma^0$ for an anti-spinor. As a result, parity maps $j^\mu$ into $\bar{u}(p_3)\gamma^0\gamma^\mu\gamma^0u(p_1)$. What is $\gamma^0\gamma^\mu\gamma^0$ ? For $\mu=0$ it is $(\gamma^0)^3=\gamma^0$, while for $\mu=k$ it is $\gamma^0\gamma^k\gamma^0=-\gamma^k$. Overall, we find that $j^\mu=(j^0,j^k)$ is mapped into $(j^0,-j^k)$ under $P$. The same statement holds for the quark current $k^\nu$, since it is again a *vector* current. The interaction is $j^\mu\eta_{\mu\nu}k^\nu=(j^0,\vec{j})(k^0,\vec{k})=j\cdot k$, and $P$ maps it into $(j^0,-\vec{j})(k^0,-\vec{k})=j\cdot k$. Hence we see that parity is conserved in any QED interaction (both vertices are $V$-like). In QCD the vertices have extra factors $\lambda^a_{ji}/2$ (upstairs) and $\lambda^b_{lk}/2$ (downstairs), but $P$ acts on the (4-dimensional) spinor degree of freedom, and does nothing in color space. Hence we reach the conclusion that parity is conserved in QCD, too.

Next it is important to recap (from Phy 2) the difference between scalars and pseudo-scalars, as well as the difference between vectors and axial-vectors. In space the parity $P$ flips the sign of any vector, e.g. $\vec{x}\mapsto-\vec{x}$ and $\vec{p}\mapsto-\vec{p}$. In space the parity $P$ keeps the sign of any axial-vector (generated as the cross-product of two vectors), e.g. $\vec{L}\equiv\vec{x}\wedge\vec{p}\mapsto(-\vec{x})\wedge(-\vec{p})=\vec{x}\wedge\vec{p}=\vec{L}$. Another example of a vector is $\vec{E}$, other axial-vectors include $\vec{B},\vec{S},\vec{\mu}$. Scalars emerge from $\vec{v}\cdot\vec{v}$ and $\vec{a}\cdot\vec{a}$, e.g. $\vec{P}^2$ or $m^2$. Pseudo-scalars emerge from $\vec{v}\cdot\vec{a}$ and $\vec{a}\cdot\vec{v}$, e.g. helicity $H\equiv\vec{S}\cdot\vec{P}$.

Finally, the experiment which showed that these considerations are actually needed is the famous Wu experiment (1956/57). If parity is conserved, there cannot be any difference between a decay product being emitted parallel/anti-parallel to an external $B$-field. Wu and collaborators showed that this is what actually happens. They considered the decay ${}^{60}\mathrm{Co} \longrightarrow {}^{60}\mathrm{Ni}^* \, e^- \, \bar{\nu}_e$, where the resulting nucleus is bound in the sample, and only the emitted $e^-$ is observed, while the $\bar{\nu}_e$ is unobserved. On a deeper level the decay is $n \longrightarrow p \, e^- \, \bar{\nu}_e$, i.e. nuclear $\beta$-decay. From a particle physics perspective the decay is $d \longrightarrow u \, e^- \, \bar{\nu}_e$.

Take a look at Fig. 11.2 in the book. Under the effect of $P$ the magnetic field $\vec{B}$ and the magnetic moment $\vec{\mu}$ stay invariant, while the momentum $\vec{p}$ of the emitted electron changes sign. Hence, if parity is conserved, an emission parallel to $\vec{\mu} \parallel \vec{B}$ must be observed equally often as an emission antiparallel to this direction (the alignment of $\vec{\mu}$ and $\vec{B}$ comes from cobalt being ferromagnetic). In other words, in a polar coordinate system with $\vartheta$ measured relative to $\vec{B}$ the emission rate must be a function which is symmetric under $\vartheta \longleftrightarrow \pi - \vartheta$.

The experiment carried out by Wu and collaborators showed a very strong asymmetry – contrary to what (almost) all physicists expected. Today it is clear that she would have deserved the Nobel prize for this discovery. The challenge at the time (and for us now) is to figure the underlying Feynman rules. What do the question marks in the fundamental fermion-fermion-$W^\pm$ vertex and the $W$-propagator at the beginning of this section stand for ?

## 8.2 Pure V-A hypothesis of charged-current weak interactions

Lorentz-invariant objects can be formed from fermion bilinears (also $\psi = \chi$ is legal) in five different ways. They have different degrees of freedom and thus correspond to different spin (of the gauge boson to which the bilinear couples). The current densities $\bar{\psi}\gamma^\mu\chi$ considered so far generate "vector-like gauge theories" (with spin-1 gauge boson). The current densities $\bar{\psi}\gamma^\mu\gamma^5\chi$ generate "axial-vector-like gauge theories" (with spin-1 gauge boson).

| # | | name | bilinear | d.o.f. | spin |
|---|---|---|---|---|---|
| $(i)$ | (S) | scalar | $\bar{\psi}\chi$ | 1 | 0 |
| $(ii)$ | (V) | vector | $\bar{\psi}\gamma^\mu\chi$ | 4 | 1 |
| $(iii)$ | (T) | tensor | $\bar{\psi}[\gamma^\mu, \gamma^\nu]\chi$ | 6 | 2 |
| $(iv)$ | (A) | axial-vector | $\bar{\psi}\gamma^\mu\gamma^5\chi$ | 4 | 1 |
| $(v)$ | (P) | pseudo-scalar | $\bar{\psi}\gamma^5\chi$ | 1 | 0 |

Around the time of the Wu experiment, some physicists suspected that charged-current weak interactions are mediated by a (possibly massive) spin-1 gauge boson (to be called $W^\pm$). According to the table, any linear combination of $(ii)$ and $(iv)$ is a candidate vertex for a $DUW$-vertex or a $\ell\nu W$ vertex. We will show that a clever combination of the two can generate maximal $P$-violation. For this purpose let us rewrite the generalized fermion current

$$\begin{aligned} j^\mu &= \bar{u}(p')\Big[g_V\gamma^\mu + g_A\gamma^\mu\gamma^5\Big]u(p) \\ &= g_V j_V^\mu + g_A j_A^\mu \qquad \text{with} \qquad j_V^\mu \equiv \bar{u}(p')\gamma^\mu u(p) \quad \text{and} \quad j_A^\mu \equiv \bar{u}(p')\gamma^\mu\gamma^5 u(p) \end{aligned} \tag{8.1}$$

and work out the chiral structure of charged weak interactions. Under $P$ the $\mu = k$ components of $j_V$ flip sign, and the $\mu = 0$ component of $j_A$. The effect of $P$ on the product $j^\mu\eta_{\mu\nu}j^\nu$ is thus

$$P: \quad j \cdot j \quad \mapsto \quad g_V^2 j_V \cdot j_V - g_V g_A[2 j_V \cdot j_A] + g_A^2 j_A \cdot j_A \tag{8.2}$$

and the relative strength of $P$-violation versus $P$-conservation is given by the ratio of the middle term to the other two terms. Hence, the requirement of *maximal parity violation* amounts to

$$\frac{2g_V g_A}{g_V^2 + g_A^2} \overset{(!)}{\cong} 1 \,. \tag{8.3}$$

Accordingly, the Feynman rule to capture the Wu experiment requires $|g_V| = |g_A|$. This leaves two options, a pure V-A theory ($j^\mu \equiv j_V^\mu - j_A^\mu$) or a pure V+A theory ($j^\mu \equiv j_V^\mu + j_A^\mu$). Theory alone cannot bring a decision, further experimental input is needed. The result was that the former option is realized in nature. A convention is the prefactor $g_W/\sqrt{2} \equiv g_{\mathrm{wk}}/\sqrt{2}$, so

$$\boxed{-\mathrm{i}\frac{g_W}{\sqrt{2}}\gamma^\mu \frac{1}{2}(1-\gamma^5) = -\mathrm{i}\frac{g_W}{\sqrt{2}}\gamma^\mu P_L} \tag{8.4}$$

is the vertex factor relevant in charged-current weak interactions, whereupon

$$\boxed{j_W^\mu \equiv \frac{g_W}{\sqrt{2}}\bar{u}(p')\gamma^\mu \frac{1}{2}(1-\gamma^5)u(p)} \tag{8.5}$$

is the charged-current bilinear. Note that the neutral-current bilinear $j_Z^\mu$ (to be discussed later) is more complicated, it will retain the $g_V, g_A$ factors, since it is *not* a pure V-A current.

## 8.3 Chiral structure of charged-current weak interactions

The pure V-A structure of the current (8.5) brings a simplification/reduction of possible interaction terms. We begin with the chiral decomposition (each of $u, u_R, u_L$ is normalized to $\sqrt{2E}$)

$$u = \underbrace{\frac{1}{2}(1+\gamma^5)}_{P_R} u + \underbrace{\frac{1}{2}(1-\gamma^5)}_{P_L} u \equiv a_R u_R + a_L u_L \tag{8.6}$$

and list the contributions which do (or do not) survive for charged-current interactions:

$$\begin{array}{llllll} \text{QED:} & \bar{u}_R\gamma^\mu \quad u_R \neq 0\,, & \bar{u}_L\gamma^\mu \quad u_R = 0\,, & \bar{u}_R\gamma^\mu \quad u_L = 0\,, & \bar{u}_L\gamma^\mu \quad u_L \neq 0 \\ \text{V-A:} & \bar{u}_R\gamma^\mu P_L u_R = 0\,, & \bar{u}_L\gamma^\mu P_L u_R = 0\,, & \bar{u}_R\gamma^\mu P_L u_L = 0\,, & \bar{u}_L\gamma^\mu P_L u_L \neq 0 \end{array} \tag{8.7}$$

In other words, only L-chirality spinors participate in interactions with $W$-bosons, i.e. just $\bar{u}_L\gamma^\mu P_L u_L$. We could do the same exercise with anti-particle spinors, and the conclusion is that only $\bar{v}_R\gamma^\mu P_L v_R$ gives a non-vanishing contribution [recall that $\frac{1}{2}(1-\gamma^5)v = v_R$].

In the ultra-relativistic limit chirality boils down to helicity. Hence, in the ultra-relativistic limit (and only then) one can draw an intuitive picture of charged-current weak interactions, see Fig. 11.4 in the book. The $e^-$ is strictly L-handed, the $e^+$ strictly R-handed, and the $\nu_e$ is strictly L-handed and the $\bar{\nu}_e$ strictly R-handed. In general *particles are strictly L-handed* and *antiparticles are strictly R-handed* if they participate in a $W$-exchange. This is also shown in Fig. 11.5 of the book, where the left panel gives the situation in the Wu experiment, and the right panel depicts the forbidden situation which never occurred in that experiment.

In fact, this fills in the missing piece of information (V-A versus V+A) mentioned above. If charged-current weak interactions were mediated by a pure V+A gauge theory, it would be the other way around (particles R-chirality, anti-particles L-chirality). In short, the Wu experiment with an additional determination of the helicity of the outgoing $e^-$ solves the question.

## 8.4 Massive W-boson propagator and effective Fermi theory

The boson propagator in QED was $-\mathrm{i}\eta_{\mu\nu}/q^2$. In weak interactions involving a $W$ we already suspected that a massive gauge boson is involved. Hence, an obvious guess is that the propagator might change to $-\mathrm{i}\eta_{\mu\nu}/(q^2 - m_W^2)$. This guess takes the difference between $m_\gamma = 0$ and $m_W = 80.4\,\mathrm{GeV}[/c^2]$ partly into account, but not fully.

The missing part is that the photon has only 2 spin-polarizations (because it is massless, e.g. left-circular and right-circular). By contrast the $W$ has 3 spin-polarizations (because it is massive, it must allow for $s_z \in \{1, 0, -1\}$ if it propagates in $z$-direction). In QED we had the completeness relation $\sum_\lambda \epsilon_\mu^{*\lambda}\epsilon_\nu^\lambda = -\eta_{\mu\nu}$, where the sum is over 2 terms. In QFD the analogous relation is $\sum_\lambda \epsilon_\mu^{*\lambda}\epsilon_\nu^\lambda = -\eta_{\mu\nu} + q_\mu q_\nu/m_W^2$, where the sum is over 3 terms. Hence

$$\boxed{\frac{-\mathrm{i}}{q^2 - m_W^2}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{m_W^2}\right)} \tag{8.8}$$

is the correct propagator for the massive $W$ boson. Note that for $q^2 \ll m_W^2$ the second term in the parentheses becomes sub-dominant, and one should expand (under these circumstances) the first term in powers of $q^2/m_W^2$. Hence a crude approximation is

$$\frac{-\mathrm{i}\eta_{\mu\nu}}{q^2 - m_W^2} \quad \longrightarrow \quad \frac{\mathrm{i}\eta_{\mu\nu}}{m_W^2} \tag{8.9}$$

if the momentum transferred is small compared to $m_W \simeq 80.4\,\mathrm{GeV}$.

In nuclear physics there is the Fermi theory as effective low-energy theory, it is a current-current (dim=6) interaction, without a gauge propagator in between. In Fig. 11.6 of the book the effective shrinking of the interaction, due to this approximation, is indicated. Comparing

$$\begin{aligned}
M_{fi} &= -\left(\frac{g_W}{\sqrt{2}}\right)^2 [\bar{u}(p_3)\gamma^\mu \frac{1}{2}(1-\gamma^5)u(p_1)] \frac{\eta_{\mu\nu} - q_\mu q_\nu/m_W^2}{q^2 - m_W^2} [\bar{u}(p_4)\gamma^\nu \frac{1}{2}(1-\gamma^5)u(p_2)] \\
M_{fi} &= \frac{G_\mathrm{F}}{\sqrt{2}}\eta_{\mu\nu}[\bar{u}(p_3)\gamma^\mu(1-\gamma^5)u(p_1)][\bar{u}(p_4)\gamma^\nu(1-\gamma^5)u(p_2)]
\end{aligned} \tag{8.10}$$

to each other, we see there is approximate agreement (for $q^2 \ll m_W^2$) if we identify

$$\boxed{\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}} \tag{8.11}$$

and we finally see where the fact that $G_F$ has dimension length-squared actually comes from.

An important application of the Fermi theory concerns the strength of the charged-current weak interaction. The decay $\mu^- \to e^-\nu_\mu\bar{\nu}_e$ is well described by the Fermi 4-fermion interaction, since $q^2 \ll m_W^2$ (why ?). Performing a non-trivial phase-space integration yields

$$\frac{1}{\tau_\mu} \simeq \Gamma_{\mu\to e\nu\bar{\nu}} = \frac{G_F^2 m_\mu^5}{192\pi^3} \tag{8.12}$$

where we used that this is the only relevant decay channel. The muon mass and lifetime are known, $m_\mu = 0.105\,658\,371\,5(35)$ GeV and $\tau_\mu = 2.196\,981\,1(22)\,10^{-6}$ s, respectively, hence

$$G_F = 1.166\,378\,7(6)\,10^{-5}\,\mathrm{GeV}^{-2}\ . \tag{8.13}$$

It is important to notice that this relation, together with (8.11) tells us that the weak coupling $g_W \equiv g_{\rm wk}$ is *not weak*. It yields the dimensionless weak coupling constant

$$\boxed{\alpha_W \equiv \frac{g_W^2}{4\pi} = \frac{8m_W^2 G_F}{4\sqrt{2}\pi}} \tag{8.14}$$

for which the numerical value of $G_F$, together with $m_W = 80.4\,\mathrm{GeV}$, yields $\alpha_W \simeq 0.034 \simeq 1/30$. In other words, the name "weak interactions" is somewhat misleading. It is not the coupling strength which makes this fundamental force weak, but rather the factor $1/(8m_W^2)$ in (8.11), which reflects the presence of the massive $W$-boson propagator at low $q^2$.

The same fact is also highlighted in the opposite limit $q^2 \gg m_Z^2$. In this high-energy limit the second line of (8.10) is *not a good approximation* to the first line. In this case one would rather ignore the $\eta_{\mu\nu}$ in the numerator and $m_W^2$ in the denominator. Hence

$$M_{fi} \simeq -\Big(\frac{g_W}{\sqrt{2}}\Big)^2 [\bar{u}(p_3)\gamma^\mu \frac{1}{2}(1-\gamma^5)u(p_1)] \frac{-q_\mu q_\nu/m_W^2}{q^2} [\bar{u}(p_4)\gamma^\nu \frac{1}{2}(1-\gamma^5)u(p_2)] \tag{8.15}$$

has a structure similar to what the QED Feynman rules would yield, except for $\eta_{\mu\nu} \to -q_\mu q_\nu/m_W^2$ in the photon propagator, the $P_L = \frac{1}{2}(1-\gamma^5)$ projectors, and an extra factor $\frac{1}{2}$ convention. In this limit the electro-magnetic and charged-weak interactions look similar, and they have a comparable magnitude (up to a factor $30/137 \simeq 0.22$). The natural place to test this assertion is in high-energy neutrino-nucleon versus electron-nucleon scattering.

Of course, at this point the alert student will ask himself/herself: What about running ? Is $\alpha_{\rm wk}$ constant as a function of $q^2$, contrary to $\alpha_{\rm em}(q^2)$ and $\alpha_{\rm st}(q^2)$ ? And in case it runs, will it grow with $q^2$ (like the former) or decrease with $q^2$ (like the latter) ? The answer is that the 1-loop formula (7.24) still holds true, but we need to replace $N_c \to 2$. Hence

$$\alpha_{\rm wk}(q^2) \simeq \frac{\alpha_{\rm wk}(\mu^2)}{1 + [11 - N_f]\alpha_{\rm wk}(\mu^2)\ln(q^2/\mu^2)/(6\pi)} \tag{8.16}$$

predicts (qualitatively) the same behavior as seen in Fig. 10.14 of the book for QCD, if $11 > N_f$. Here, $N_f$ is the number of fermion-pairs that couple to the $W$-current, e.g. $e^-\bar{\nu}_e$, $\mu^-\bar{\nu}_\mu$, $\tau^-\bar{\nu}_\tau$ in the lepton sector, and $u\bar{d}$, $c\bar{s}$, $t\bar{b}$ in the quark-sector (here things get more involved through CKM mixing). Hence in the SM $N_f = 6$, and the square bracket is positive, like in QCD.

## 8.5 Lepton universality

An important concept in the SM is *lepton universality*, i.e. all lepton current-pairs couple with the same strength to the $W$-boson. To understand it let us compare $\mu$-decay to $\tau$-decay.

The leading-order Feynman diagram for $\mu$-decay is shown in Fig. 12.1 of the book (the remainder is insignificant). The Feynman rule bring a factor $-ig_W/\sqrt{2}$ at both the muon and the electron vertex, and this yields (8.12). In the event the two couplings would be different we would have this formula with $G_F^2 \to G_F^{(\mu)}G_F^{(e)}$ replaced, hence

$$\Gamma_{\mu\to e\nu_\mu\bar{\nu}_e} = \frac{G_F^{(\mu)}G_F^{(e)}m_\mu^5}{192\pi^3} + ... \tag{8.17}$$

where the dots denote insignificant decay channels. The leading-order Feynman diagrams for $\tau$-decay are shown in Fig. 12.2 of the book (the remainder is insignificant). Since these are distinguishable alternatives not the amplitudes are added, but their absolute-squares, i.e. the partial widths. Hence the total width of the $\tau$ decomposes as

$$\Gamma_\tau \equiv \Gamma_{\tau\to\text{anything}} = \Gamma_{\tau\to\mu\nu_\tau\bar{\nu}_\mu} + \Gamma_{\tau\to e\nu_\tau\bar{\nu}_e} + \Gamma_{\tau\to\nu_\tau d\bar{u}} + ... \tag{8.18}$$

and the total width is the inverse of the $\tau$-lifetime, $\Gamma_{\tau\to\text{anything}} = 1/\tau_\tau$. For the first and the second term a similar decomposition of the would-be universal $G_F^2$ holds true, for instance

$$\Gamma_{\tau\to e\nu_\tau\bar{\nu}_e} = \frac{G_F^{(\tau)} G_F^{(e)} m_\tau^5}{192\pi^3} + ... \tag{8.19}$$

Hence, measuring $\tau_\tau$ and $\tau_\mu$ as well as the *branching ratio*

$$B(\tau^- \to e^-\bar{\nu}_e\nu_\tau) \equiv \frac{\Gamma(\tau^- \to e^-\bar{\nu}_e\nu_\tau)}{\Gamma_\tau} \tag{8.20}$$

yields $G_F^{(\tau)}/G_F^{(\mu)} = 1.0023(33)$ which is well consistent with the assumption of the SM that there is a universal $G_F$ for leptonic charged-current weak interactions. The hypothesis that this feature might not hold true if measurements are made (significantly) more precise has recently drawn attention. And (some) theorists are quick at "explaining" yet-unobserved (or at least not-yet firmly established) phenomena in specific "beyond-standard-model theories".

## 8.6 Summary

- There are two types of weak interactions: charged-current versus neutral-current.
- Charged-current interactions happen through $W^\pm$ exchange, neutral-current through $Z^0$.
- Charged-current (CC) weak interactions have pure V-A structure: $-\mathrm{i}\frac{g_W}{\sqrt{2}}\gamma^\mu\frac{1}{2}[1-\gamma^5]$.
- Neutral-current (NC) weak interactions have complicated structure: $-\mathrm{i}\frac{g_W}{\cos(\theta_W)}\gamma^\mu\frac{1}{2}[c_V - c_A\gamma^5]$.
- For any $m$ the charged-current couples to L-chirality particles and R-chirality anti-particles.
- For $m \to 0$ the charged-current couples to L-helicity particles and R-helicity anti-particles.
- In technical terms only $\bar{u}_L\gamma^\mu P_L u_L$ and $\bar{v}_R\gamma^\mu P_L v_R$ yield non-zero contributions.
- All leptons couple with the same strength $-\mathrm{i}g_W/\sqrt{2}$ to the $W$-boson ("lepton universality").

# 9 Neutrino flavors and MNS matrix

## 9.1 Old versus new SM

As an update to the "old SM", formulated in the 1970 decade, the present "new SM" emerged in the 1990 and 2000 decades. In the former theory neutrinos are massless; in the latter theory neutrinos are Dirac states with $m > 0$. As a result, the neutrino flavors $|\nu_e\rangle$, $|\nu_\mu\rangle$, $|\nu_\tau\rangle$ mix into each other, and the "new SM" has 7 more parameters (see below) than the "old SM".

Neutrinos participate only in charged-current weak interactions, i.e. there is no ionization in a detector. The $|\nu_e\rangle$ is defined as the state associated with the $e^+$ in a $W^+$ decay, the $|\nu_\mu\rangle$ is defined as the state associated with the $\mu^+$ in a $W^+$ decay, and the $|\nu_\tau\rangle$ is defined as the state associated with the $\tau^+$ in a $W^+$ decay. These are flavor-eigenstates, but not mass eigenstates. By contrast, the mass-eigenstates $|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle$ have masses $m_1, m_2, m_3$, respectively, but they have no lepton numbers, since they are linear superpositions of the flavor-eigenstates.

Originally (i.e. prior to the "old SM") people thought there is only one neutrino which couples to all leptons, so only $L$ is conserved. If that were true, the process $\mu \to e\gamma$ could take place through a 1-loop diagram, see Fig. 13.2 in the book, with a sizable amplitude. The process has never been seen, though. In the "old SM" the lepton numbers $L_e, L_\mu, L_\tau$ are generationwise conserved, so this amplitude is zero. In the "new SM" the lepton numbers $L_e, L_\mu, L_\tau$ are generationwise conserved at the vertices, but the amplitude for $\mu \to e\gamma$ is non-zero (albeit unmeasurably small for decades to come), since this process works only through mixing.

The notion that $L_e, L_\mu, L_\tau$ are separately conserved emerged from short-distance experiments. Take a look at Fig. 13.1 in the book; it shows the setup of the famous Columbia 2-neutrino experiment. The neutrino is produced in association with an $e^+$. If there was only one neutrino flavor, one would see $e^-, \mu^-, \tau^-$ in about equal numbers (in the high-energy limit where their masses can be treated as negligible). As only $e^-$ were seen, the hypothesis emerged that an $\nu_e$ is produced (rather than a generic $\nu$) which cannot decay into an $\mu^-$ or $\tau^-$.

The difference to the "new SM" is that in the latter theory only the sum $L \equiv L_e + L_\mu + L_\tau$ is conserved, not $L_e, L_\mu, L_\tau$ individually. In other words, the neutrino flavor $\nu_e$ can oscillate, in flight, to $\nu_\mu$ or $\nu_\tau$. This implies that one must distinguish between mass eigenstates, to be called $\nu_1, \nu_2, \nu_3$, and flavor eigenstates, to be named $\nu_e, \nu_\mu, \nu_\tau$. The latter states have no "mass".

## 9.2 Solar neutrinos

For a long time there was a "solar neutrino deficit", i.e. fewer $\nu_e$ seemed to arrive on earth than expected. The point is that the overall power release of the sun is well known. If the "solar standard model" is correct, the sun produces a known distribution of (electron) neutrinos from the *pp*-cycle, as well as known $^7$Be and *pep* peaks, and a known $^8$B shoulder. The expected kinematic distribution of solar $|\nu_e\rangle$ is shown as Fig. 13.3 in the book. One should add that detecting neutrinos with $E = O(10\,\mathrm{MeV})$ is easier than with $E = O(1\,\mathrm{MeV})$ or $E = O(100\,\mathrm{keV})$, so the difficulty increases in the figure noticeably from the right to the left.

The Homestake experiment ($\nu_e + {}^{37}_{17}\mathrm{Cl} \to {}^{37}_{18}\mathrm{Ar} + e^-$, radiochemical) found fewer $\nu_e$ in the $^8$B shoulder than expected. The Superkamiokande experiment (Cherenkov light in $H_2O$) found fewer $\nu_e$ in the $^8$B shoulder than expected; the new ingredients are the directional information and that some $\nu_\mu$ are actually seen. The Sudbury Neutrino Observatory SNO (Cherenkov light in $D_2O$) was a significant improvement over Superkamiokande. The reason can be understood with Fig. 13.7 in the book. All neutrino species interact through neutral-current (NC) processes, i.e. via $Z$-exchange (see right panel), and may thus trigger a $d$-breakup. Due to the relatively small binding energy of the deuteron (2.2 MeV), the charged-current (CC) process (via $W$-mediated $\nu_e d \to epp$, see left panel) works only for $\nu_e$, but not for $\nu_\mu, \nu_\tau$. By contrast, the neutral-current (NC) process (via $Z$-mediated $\nu_\ell d \to \nu_\ell np$, see right panel) is allowed for all $\nu_\ell$. In addition, there is elastic scattering (ES) at the atomic electrons, see Fig. 13.5, which also generates an asymmetry between $\nu_e$ and $\nu_\mu, \nu_\tau$ (see book for details). As a result SNO found

$$\begin{aligned} \text{flux}(\nu_e) &= 1.76(10) \cdot 10^{-6}\,\text{cm}^{-2}\,\text{s}^{-1} \\ \text{flux}(\nu_\mu) + \text{flux}(\nu_\tau) &= 3.41(63) \cdot 10^{-6}\,\text{cm}^{-2}\,\text{s}^{-1} \end{aligned} \tag{9.1}$$

to be compared to the expected $\text{flux}(\nu_e) = 5.1(9) \cdot 10^{-6}\,\text{cm}^{-2}\,\text{s}^{-1}$ in the "solar standard model".

What catches our attention is that the two numbers in (9.1), when *added*, match the expectation pretty well. In addition the first line of (9.1) is perfectly consistent with the results of Homestake and Superkamiokande. A natural hypothesis is thus that $\nu_e$ are produced at the expected rate in the core of the sun, but *oscillate* into $\nu_\mu, \nu_\tau$ on their way to the earth.

## 9.3 Weak versus mass eigenstates

The Hamiltonian has eigenstates $|\nu_i\rangle$ $(i = 1, 2, 3)$ which, at any given time, relate to the weak eigenstates $|\nu_\alpha\rangle$ $(\alpha = e, \mu, \tau)$ unitarily. At any fixed $t$ the relation is

$$|\nu_\alpha\rangle = \sum_{i=1}^{3} U_{\alpha i} |\nu_i\rangle \qquad \text{or} \qquad |\nu_i\rangle = \sum_\alpha (U^\dagger)_{i\alpha} |\nu_\alpha\rangle = \sum_\alpha U^*_{\alpha i} |\nu_\alpha\rangle \ . \tag{9.2}$$

Solar neutrinos have energies $O(1\,\text{MeV})$; in comparison their rest-masses are small (from cosmology one has the bound $\sum_{i=1}^3 m_i < 0.2\,\text{eV}$; the concept of mass is mis-conceived for $\nu_e, \nu_\mu, \nu_\tau$). As a result neutrinos propagate ultra-relativistically. Feynman rules can be formulated in terms of flavor-eigenstates ("standard") or mass-eigenstates ("alternative").

- Standard Feynman rules ($\nu_\alpha$ immediately starts oscillating, $\alpha \in \{e, \mu, \tau\}$, see Fig. 13.11):
  $\nu_\alpha \to \ell_\alpha^- W$ vertex is $-\mathrm{i}\frac{g_W}{\sqrt{2}} \bar{u}(\ell_\alpha)\gamma^\mu P_L u(\nu_\alpha)$
  $\ell_\alpha^+ \to \bar{\nu}_\alpha W$ vertex is $-\mathrm{i}\frac{g_W}{\sqrt{2}} \bar{v}(\ell_\alpha)\gamma^\mu P_L v(\nu_\alpha)$
  $W \to \ell_\alpha^- \bar{\nu}_\alpha$ vertex is $-\mathrm{i}\frac{g_W}{\sqrt{2}} \bar{u}(\ell_\alpha)\gamma^\mu P_L v(\nu_\alpha)$

- Alternative Feynman rules ($\nu_i$ stays till next vertex, $i \in \{1, 2, 3\}$, see Fig. 13.10):
  $\nu_k \to \ell_\alpha^- W$ vertex is $-\mathrm{i}\frac{g_W}{\sqrt{2}} \bar{u}(\ell_\alpha)\gamma^\mu P_L U_{\alpha k} u(\nu_k)$
  $\ell_\alpha^+ \to \bar{\nu}_k W$ vertex is $-\mathrm{i}\frac{g_W}{\sqrt{2}} \bar{v}(\ell_\alpha)\gamma^\mu P_L U_{\alpha k} v(\nu_k)$
  $W \to \ell_\alpha^- \bar{\nu}_k$ vertex is $-\mathrm{i}\frac{g_W}{\sqrt{2}} \bar{u}(\ell_\alpha)\gamma^\mu P_L U_{\alpha k} v(\nu_k)$
  All of these have the structure $\bar{x}(\ell_\alpha)...y(\nu_k)$ with $U_{\alpha k}$ and $x, y \in \{u, v\}$.

  $\bar{\nu}_k \to \ell_\alpha^+ W$ vertex is $-\mathrm{i}\frac{g_W}{\sqrt{2}} \bar{v}(\nu_k)\gamma^\mu P_L U^*_{\alpha k} v(\ell_\alpha)$
  $\ell_\alpha^- \to \nu_k W$ vertex is $-\mathrm{i}\frac{g_W}{\sqrt{2}} \bar{u}(\nu_k)\gamma^\mu P_L U^*_{\alpha k} u(\ell_\alpha)$
  $W \to \ell_\alpha^+ \nu_k$ vertex is $-\mathrm{i}\frac{g_W}{\sqrt{2}} \bar{u}(\nu_k)\gamma^\mu P_L U^*_{\alpha k} v(\ell_\alpha)$
  All of these have the structure $\bar{x}(\nu_k)...y(\ell_\alpha)$ with $U^*_{\alpha k}$ and $x, y \in \{u, v\}$.

## 9.4 Neutrino oscillation with 2 flavors

The mixing is described by a matrix $U \in SU(2)$, and $SU(2)$ has $8-4-1=3$ parameters, but 2 can be absorbed into fields, so 1 relevant parameter remains. Thus we consider

$$\begin{aligned}|\nu_e(0)\rangle &= \phantom{-}|\nu_1(0)\rangle \cos\theta + |\nu_2(0)\rangle \sin\theta \\ |\nu_\mu(0)\rangle &= -|\nu_1(0)\rangle \sin\theta + |\nu_2(0)\rangle \cos\theta \end{aligned} \tag{9.3}$$

where the mixing angle $\theta$ is a fundamental constant (like $\theta_{\rm C}$ in quark physics, see later), and

$$\begin{aligned}|\nu_1(0)\rangle &= |\nu_e(0)\rangle \cos\theta - |\nu_\mu(0)\rangle \sin\theta \\ |\nu_2(0)\rangle &= |\nu_e(0)\rangle \sin\theta + |\nu_\mu(0)\rangle \cos\theta \end{aligned} \tag{9.4}$$

is the inverse relation. Any mass eigenstate (ES) evolves, for $\vec{p} = p\vec{e}_z$, through the QM phase

$$|\nu_i(t)\rangle = |\nu_i(0)\rangle\, e^{-{\rm i}[Et-pz]} \tag{9.5}$$

with $z \simeq t$ and $E - p = \sqrt{p^2+m_i^2} - p \simeq p(1+\frac{m_i^2}{2p^2}) - p = \frac{m_i^2}{2p}$, so

$$|\nu_i(t)\rangle \simeq |\nu_i(0)\rangle\, e^{-{\rm i}m_i^2 t/(2p)}\ . \tag{9.6}$$

As a result we have

$$\begin{aligned}|\nu_e(t)\rangle &= \phantom{-}|\nu_1(0)\rangle e^{-{\rm i}m_1^2 t/(2p)} \cos\theta + |\nu_2(0)\rangle e^{-{\rm i}m_2^2 t/(2p)} \sin\theta \\ |\nu_\mu(t)\rangle &= -|\nu_1(0)\rangle e^{-{\rm i}m_1^2 t/(2p)} \sin\theta + |\nu_2(0)\rangle e^{-{\rm i}m_2^2 t/(2p)} \cos\theta \end{aligned} \tag{9.7}$$

and the amplitude for $\nu_e \to \nu_e$ is given by

$$\begin{aligned}A_{\nu_e\to\nu_e}(t) &\equiv \langle\nu_e|\nu_e(t)\rangle \\ &= \langle\nu_e|\nu_1(0)\rangle e^{-{\rm i}m_1^2 t/(2p)} \cos\theta + \langle\nu_e|\nu_2(0)\rangle e^{-{\rm i}m_2^2 t/(2p)} \sin\theta \\ &= e^{-{\rm i}m_1^2 t/(2p)} \cos^2\theta + e^{-{\rm i}m_2^2 t/(2p)} \sin^2\theta \end{aligned} \tag{9.8}$$

where in the last step relation (9.4) was plugged in, and the orthogonality of $|\nu_1\rangle, |\nu_2\rangle$ was used. Taking the modulus squared yields the "survival probability"

$$\begin{aligned}P_{\nu_e\to\nu_e}(t) &= \cos^4\theta + 2\cos^2\theta\sin^2\theta\cos\Big(\frac{(m_2^2-m_1^2)t}{2p}\Big) + \sin^4\theta \\ &= 1 - \sin^2(2\theta)\sin^2\Big(\frac{\delta m^2\, t}{2p}\Big)\end{aligned} \tag{9.9}$$

where $\delta m^2 \equiv |m_2^2 - m_1^2|$ is the gap between the squared masses.

An analogous calculation for $\nu_e \to \nu_\mu$ or unitarity yield the "oscillation probability"

$$P_{\nu_e\to\nu_\mu}(t) = \sin^2(2\theta)\sin^2\Big(\frac{\delta m^2\, t}{2p}\Big) \tag{9.10}$$

and we learn: ($i$) the oscillation probability knows about $\delta m^2 \equiv |m_2^2 - m_1^2|$, not about the individual masses (even the hierarchy does not matter), ($ii$) the oscillation time/length scales inversely with the neutrino energy (here $p$). A rewrite in terms of the oscillation length yields

$$P_{\nu_e\to\nu_\mu}(L) = \sin^2(2\theta)\sin^2\Big(1.27\,\frac{\delta m^2[{\rm eV}^2]\cdot L[{\rm m}]}{E_\nu[{\rm MeV}]}\Big) \tag{9.11}$$

since $c \simeq L/t$ and $p \simeq E$. An illustration is given in Fig. 13.12 of the book. The "transition probability" (full line) and "survival probability" (dashed line) are shown as a function of $L$/km. The former one never exceeds $\sin^2(2\theta) \simeq 0.8$, the latter one never falls below $1-\sin^2(2\theta) \simeq 0.2$. In retrospect it is clear why the Columbia 2-neutrino experiment suggested that $L_e, L_\mu, L_\tau$ might be individually conserved; the whole experiment took place at a distance $L \ll 1\,\mathrm{km}$.

Let us add that such appearance/disappearance experiments can be performed with neutrinos or anti-neutrinos at the source. In addition, any combination of electron/muon-neutrinos (in the 2-flavor case) can be used. A quick summary is given in the table.

| name | type | production process |
|---|---|---|
| sun | $\nu_e$ | from $p^+p^+ \to d^+e^+\nu_e$ |
| atmosphere | $\nu_\mu, \bar{\nu}_\mu, \nu_e, \bar{\nu}_e$ | secondary from cosmic rays |
| reactor | $\bar{\nu}_e$ | from $n \to p^+e^-\bar{\nu}_e$ |
| accelerator | $\nu_\mu, \bar{\nu}_\mu$ | from $\mu^+ \to e^+\nu_e\bar{\nu}_\mu$, $\mu^- \to e^-\bar{\nu}_e\nu_\mu$ |

## 9.5 Neutrino oscillation with 3 flavors

The mixing is described by a matrix $U \in SU(3)$, and $SU(3)$ has $18-9-1=8$ parameters, but 4 can be absorbed into fields, so 4 relevant parameters remain. Thus we consider

$$\underbrace{\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \\ |\nu_\tau\rangle \end{pmatrix}}_{\text{weak ES}} = \underbrace{\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix}}_{U_{\mathrm{MNS}}} \underbrace{\begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \\ |\nu_3\rangle \end{pmatrix}}_{\text{mass ES}} \tag{9.12}$$

and the letters (P)MNS stand for Pontecorvo-Maki-Nakagawa-Sakata who introduced it.

There is a zillion ways to parametrize an $SU(3)$ matrix. The "standard parametrization"

$$\begin{aligned} U_{\mathrm{MNS}} &= \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{31} & & -s_{31}e^{-\mathrm{i}\delta} \\ & 1 & \\ s_{31}e^{\mathrm{i}\delta} & & c_{31} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{31} & s_{12}c_{31} & s_{31}e^{-\mathrm{i}\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{31}e^{\mathrm{i}\delta} & c_{12}c_{23}-s_{12}s_{23}s_{31}e^{\mathrm{i}\delta} & s_{23}c_{31} \\ s_{12}s_{23}-c_{12}c_{23}s_{31}e^{\mathrm{i}\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{31}e^{\mathrm{i}\delta} & c_{23}c_{31} \end{pmatrix} \end{aligned} \tag{9.13}$$

is the most useful/common one, and it will be reused in the CKM context (i.e. in quark physics) below. Note that it has 3 angles ($\theta_{12}, \theta_{23}, \theta_{31}$) and one phase ($\delta_{\mathrm{MNS}}$); the abbreviations used are $c_{12} \equiv \cos\theta_{12}$, $s_{12} \equiv \sin\theta_{12}$, and similar for $\theta_{23}, \theta_{31}$. In this parametrization only the upper-right $1\times1$ entry and the lower-left $2\times2$ block are xomplex, but this is a convention.

A typical "survival probability" picture with 3 flavors is shown in Fig. 13.18 of the book. There are two amplitudes; $\sin^2(2\theta_{12}) \simeq 0.1$ governs the smaller (short-time/scale) oscillation, $\sin^2(2\theta_{31}) \simeq 0.8$ governs the larger (long-time/scale) oscillation. By tracing out such oscillations, one can determine the angles $\theta_{12}, \theta_{23}, \theta_{31}$ from the oscillation maxima/minima. In close analogy to eqn. (9.10) in the 2-flavor case, the oscillation periods yield information on the gaps $m_2^2-m_1^2$, $m_3^2-m_1^2$, $m_3^2-m_2^2$ (of which only two are independent).

Let us comment on two physics issues. The phase $\delta \equiv \delta_{\mathrm{MNS}}$ is one of two sources of CP-violation in the SM (the phase $\delta_{\mathrm{CKM}}$ in the quark sector will be discussed below). The latter phase is known quite accurately, the former one less so. It seems unlikely that these two sources of CP-violation can (together) explain the strength of the baryon asymmetry in the universe (cf. Sacharov criteria below), but the last word on this is issue is not yet spoken. In the event the neutrinos have a Majorana component (i.e. may oscillate into their anti-particles; this would amount to a process with $|\Delta L_e| = 2$ or $|\Delta L_\mu| = 2$ or $|\Delta L_\tau| = 2$) there would be two additional CP-violating phases, since $U_{\mathrm{MNS}}$ is right-multiplied with a matrix $\mathrm{diag}(e^{\mathrm{i}\alpha_1}, e^{\mathrm{i}\alpha_2}, 1)$.

## 9.6 Experimental results

Experimental results are desired for three masses $m_1, m_2, m_3$, three angles $\theta_{12}, \theta_{23}, \theta_{31}$, and one phase $\delta$. In total 7 quantities need to be measured.

In the 2-flavor case oscillation measurements can determine only the gap $\delta m^2 \equiv m_2^2 - m_1^2$ between the squared masses, not the individual masses $m_1, m_2$. Here we omit the modulus symbol; we simply define $m_1$ to be the lighter one of the two masses.

In the 3-flavor case the situation is similar; only the smaller gap $\delta m^2 \equiv m_2^2 - m_1^2$, as well as the larger (by absolute magnitude) gap $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$ (which can be positive or negative) can be determined. The current experimental situation can be summarized as

$$\begin{aligned} \delta m^2 &\equiv m_2^2 - m_1^2 = 7.5(2)\, 10^{-5}\,\mathrm{eV}^2 && \longrightarrow \max(m_1, m_2) \simeq 0.009\,\mathrm{eV} \\ \Delta m^2 &\equiv m_3^2 - \frac{m_1^2 + m_2^2}{2} = \pm 2.4(1)\, 10^{-3}\,\mathrm{eV}^2 && \longrightarrow \max(m_3, \dots) \simeq 0.049\,\mathrm{eV} \end{aligned} \tag{9.14}$$

and this means that the *hierarchy* of the three masses is not yet pinned down. There can be a "normal hierarchy" or an "inverted hierarchy", see Fig. 13.15 of the book.

In addition, the mixing angles and the phase have been determined (in the latter case with significant experimental uncertainty). The present state of knowledge is summarized as

$$\theta_{12} = 34(1)^\circ\ , \quad \theta_{23} = 47(2)^\circ\ , \quad \theta_{31} = 8.5(1)^\circ\ , \quad \delta = 235(35)^\circ \tag{9.15}$$

which indicates near-maximal mixing between the 2nd and 3rd generation ($\theta_{23} \sim 45^\circ$). And

$$|U_{\mathrm{MNS}}| \equiv \begin{pmatrix} |U_{e1}| & |U_{e2}| & |U_{e3}| \\ |U_{\mu 1}| & |U_{\mu 2}| & |U_{\mu 3}| \\ |U_{\tau 1}| & |U_{\tau 2}| & |U_{\tau 3}| \end{pmatrix} \simeq \begin{pmatrix} 0.85 & 0.50 & 0.17 \\ 0.35 & 0.60 & 0.70 \\ 0.35 & 0.60 & 0.70 \end{pmatrix} \tag{9.16}$$

indicates that there is no obvious hierarchy in the absolute magnitudes of the entries. In the next section we will see that $U_{\mathrm{MNS}}$ is numerically quite different from its cousin $U_{\mathrm{CKM}}$.

## 9.7 Summary

- In the pre-SM there is one (massless) neutrino; only the total lepton number $L$ is conserved.
- In the old-SM there are 3 massless neutrinos, and $L_e, L_\nu, L_\tau$ are separately conserved.
- In the new-SM the 3 neutrinos are massive and mix; due to oscillations only $L$ is conserved.
- Flavor-eigenstates $|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle$ and mass-eigenstates $|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle$ are related via $U_{\mathrm{MNS}}$.
- All neutrino physics is captured in 7 parameters (masses $m_{1,2,3}$ and $\theta_{12,23,31}, \delta_{\mathrm{MNS}}$ of $U_{\mathrm{MNS}}$).
- Squared-mass-gaps and mixing angles can be determined in oscillation experiments.
- The phases $\delta_{\mathrm{MNS}}$ and $\delta_{\mathrm{CKM}}$ are the only relevant sources of CP-violation in the SM.

# 10 Quark flavors and CKM matrix

## 10.1 CP-violation in the early universe

Astronomers spent a great deal of effort to establish that celestial bodies are made from matter and not anti-matter. This holds true in our solar system, in the systems formed by nearby stars and their planets (if any), in the milkyway, in the neighborhood of our galaxy, in distant galaxies – in fact in the whole visible universe. How could that happen ?

We tend to think that fundamental laws of physics should have a high degree of symmetry, while asymmetries are created by external (initial) conditions. Hence, the question is whether a given asymmetry is enhanced or diminished as the system evolves in time.

The baryon asymmetry of the universe (i.e. the excess of the number of baryons over anti-baryons) is linked to CP-violation in the early universe. The early phase is characterized by a period with $kT \gg M_B$, where $B \in \{p^+, n^0, ...\}$ is any baryon in the SM. As long as this condition is met, there is a forward-backward balance in the reaction $\gamma + \gamma \rightleftarrows p + \bar{p}$. But two epochs affect one or the other direction. The epoch of "decoupling" basically stops the forward reaction (see literature/wikipedia for details). The epoch of "baryogenesis" thins the density of the universe so much that even the backward reaction becomes very rare.

As a result, both $n_B$ and $n_{\bar{B}}$ are fixed to a number which implies $(n_B - n_{\bar{B}})/n_\gamma \simeq 10^{-9}$. We may think of every photon in the universe (there are about $1...2 \cdot 10^{89}$ of them) as stemming from the backward reaction mentioned above. This amounts to the statement that for $10^9$ anti-baryons there were $10^9 + 1$ baryons, and all but the 1 annihilated.

In 1967 Sakharov formulated three criteria which need to be satisfied to establish a matter anti-matter asymmetry in the universe: $(i)$ there must be $B$-violating processes $[B \equiv n_B - n_{\bar{B}}]$, $(ii)$ the discrete symmetries C and CP must be violated, $(iii)$ there must be sufficient departure from equilibrium. Recall that C is nearly maximally violated in weak interactions, so we should focus our attention on whether there is sufficient CP violation to "explain" the baryon anti-baryon asymmetry in the universe.

In the SM there are (in total) three known sources of CP-violation: $(a)$ the phase $\delta_{\mathrm{MNS}}$ in $U_{\mathrm{MNS}}$ [previous section], $(b)$ the phase $\delta_{\mathrm{CKM}}$ in $U_{\mathrm{CKM}}$ [this section], $(c)$ the angle $\theta$ in $L_{\mathrm{QCD}+\theta}$ [non-perturbative effect, empirically $\theta = 0$]. As a result, the phenomenologically relevant question is whether the sources $(a)$ and $(b)$ are (together) strong enough to explain the baryon asymmetry in the universe. This question is not completely answered yet (most experts think the answer is likely negative). If this is to evolve into an established fact, it would be a clear indication that there must be physics beyond the standard model (BSM).

## 10.2 Cabibbo angle for 2 flavors

Consider Fig. 14.1 in the book. The left panel shows the decay $\mu^- \to \nu_\mu e^- \bar{\nu}_e$, with two vertex factors $G_F^{(\mu)\,1/2}$ and $G_F^{(e)\,1/2}$ (fundamentally either one stands for $g_W/\sqrt{2}$). The right panel shows a similar diagram, except that $\mu^- \to \nu_\mu$ is replaced by the quark line $d \to u$, and $G_F^{(\mu)}$ is replaced by $G_F^{(du)}$ [in the book called $G_F^{(\beta)}$]. We could draw a similar diagram where the quark line is $s \to u$; the associate factor would be $G_F^{(su)}$. In fact, any down-type to up-type quark line would work; this yields in total 9 different factors $G_F^{(DU)}$ with $D \in \{d, s, b\}$ and $U \in \{u, c, t\}$.

In Sec. 8 we found "lepton universality", i.e. $G_F^{(e)} = G_F^{(\mu)} = G_F^{(\tau)}$ is the same for all lepton flavors. Does this feature extend to charged-current weak interactions with quark vertices? In other words, is the universal leptonic $G_F$ identical to $G_F^{(du)}$ and $G_F^{(su)}$? Experiment finds $G_F^{(e)} > G_F^{(du)} \gg G_F^{(su)}$. Hence the answer is "no", but it is interesting to note that the suppression factors, when squared, add up to 1 (in very good approximation).

A way to incorporate this is depicted in Fig. 14.2 of the book. We should equip the $d \to u$ transition with an extra factor $\cos(\theta_\mathrm{C})$, $s \to u$ with an extra factor $\sin(\theta_\mathrm{C})$, $d \to c$ with an extra factor $-\sin(\theta_\mathrm{C})$, and $s \to c$ with an extra factor $\cos(\theta_\mathrm{C})$. The angle $\theta_\mathrm{C}$ is referred to as the "Cabibbo angle". A glimpse at the quark table in Sec. 1 shows that a "vertical transition" gets an extra factor $\cos(\theta_\mathrm{C})$, while a "crossing transition" gets an extra factor $\pm\sin(\theta_\mathrm{C})$.

Taking another look at Fig. 14.2 we realize that we could combine the 1st and the 2nd panel into a single $d' \to u$ transition, if we define $|d'\rangle$ as the linear combination $\cos(\theta_\mathrm{C})|d\rangle + \sin(\theta_\mathrm{C})|s\rangle$. In a similar spirit we might combine the 3rd and the 4th panel into a single $s' \to c$ transition, if we define $|s'\rangle$ as the linear combination $-\sin(\theta_\mathrm{C})|d\rangle + \cos(\theta_\mathrm{C})|s\rangle$. In other words,

$$\underbrace{\begin{pmatrix} |d'\rangle \\ |s'\rangle \end{pmatrix}}_{\text{weak ES}} = \underbrace{\begin{pmatrix} \cos(\theta_\mathrm{C}) & \sin(\theta_\mathrm{C}) \\ -\sin(\theta_\mathrm{C}) & \cos(\theta_\mathrm{C}) \end{pmatrix}}_{\text{Cabibbo rotation}} \underbrace{\begin{pmatrix} |d\rangle \\ |s\rangle \end{pmatrix}}_{\text{mass ES}} \tag{10.1}$$

is a rotation among the first 2 generations of the $D$-quarks. Alternatively, one could have introduced the rotation among the first 2 generations of the $U$-quarks, but historically the former option was chosen (at the time the $c$-quark was not yet discovered). As an aside, please note that one can not mix quarks with unequal electric charge.

A way to measure $\theta_\mathrm{C}$ is to compare the leptonic decays $\pi^-(d\bar{u}) \to \mu^-\bar{\nu}_\mu$ and $K^-(s\bar{u}) \to \mu^-\bar{\nu}_\mu$. As can be seen from Fig. 14.3 in the book, in both cases a $W$ boson is created which subsequently decays into $\mu^-\bar{\nu}_\mu$. The left vertices of the diagrams differ by a factor $\cos(\theta_\mathrm{C})$ or $\sin(\theta_\mathrm{C})$, respectively, the right factors are both $g_W/\sqrt{2}$. As a result, we have the exclusive widths

$$\begin{aligned} \Gamma(\pi^- \to \mu^-\bar{\nu}_\mu) &\propto |\cos(\theta_\mathrm{C})\, g_W^2/2|^2 \propto \cos^2(\theta_\mathrm{C}) G_F^2 \\ \Gamma(K^- \to \mu^-\bar{\nu}_\mu) &\propto |\sin(\theta_\mathrm{C})\, g_W^2/2|^2 \propto \sin^2(\theta_\mathrm{C}) G_F^2 \end{aligned} \tag{10.2}$$

and from the ratio one obtains $\tan^2(\theta_\mathrm{C})$. The number people have in mind is

$$\cos(\theta_\mathrm{C}) = 0.97425(22) \qquad \text{or} \qquad \theta_\mathrm{C} \simeq 13^\circ \tag{10.3}$$

and this means that the mixing is small, i.e. $d'$ is predominantly $d$, and $s'$ is predominantly $s$.

Coming back to (10.1), it seems that everything is conceptually identical to the situation in neutrino physics. There the "weak ES" was $|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle$, arranged as column vector, and "mass ES" was $|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle$, see (9.12). However, the role of "flavor ES" is different; in neutrino physics the LHS is a flavor ES, in quark physics the RHS is a flavor ES.

## 10.3 CKM matrix for 3 flavors

The Cabibbo mixing for 2 generations naturally generalizes to the Kobayashi-Maskawa mixing for 3 generations. The defining equation for the CKM matrix is

$$\underbrace{\begin{pmatrix} |d'\rangle \\ |s'\rangle \\ |b'\rangle \end{pmatrix}}_{\text{weak ES}} = \underbrace{\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}}_{V_\mathrm{CKM}} \underbrace{\begin{pmatrix} |d\rangle \\ |s\rangle \\ |b\rangle \end{pmatrix}}_{\text{mass ES}} \tag{10.4}$$

and one should remember that $U$-quarks serve as row-index, and $D$-quarks serve as column index. Again, the mass eigenstates simultaneously serve as flavor eigenstates (in contradistinction to the situation in neutrino physics).

The standard Feynman rules in quark physics are in terms of the flavor=mass eigenstates, but with the CKM matrix elements as extra factors. Take a look at the upper three panels in Fig. 14.5, and augment that row by a fourth panel with the vertex $\bar{u}d \to W^-$. Then

$$\begin{aligned}
j^\mu_{\mathrm{CC}} &\equiv -\mathrm{i}\frac{g_W}{\sqrt{2}}\bar{u}(u)\gamma^\mu P_L u(d)\cdot V_{ud} \\
j^\mu_{\mathrm{CC}} &\equiv -\mathrm{i}\frac{g_W}{\sqrt{2}}\bar{v}(u)\gamma^\mu P_L v(d)\cdot V_{ud} \\
j^\mu_{\mathrm{CC}} &\equiv -\mathrm{i}\frac{g_W}{\sqrt{2}}\bar{u}(u)\gamma^\mu P_L v(d)\cdot V_{ud} \\
j^\mu_{\mathrm{CC}} &\equiv -\mathrm{i}\frac{g_W}{\sqrt{2}}\bar{v}(u)\gamma^\mu P_L u(d)\cdot V_{ud}
\end{aligned} \tag{10.5}$$

would give a summary of the effect of the CC weak Feynman rules for these four vertices. Now take a look at the lower three panels in Fig. 14.5, and augment that row by a fourth panel with the vertex $\bar{d}u \to W^+$. In this case the CC weak Feynman rules give

$$\begin{aligned}
j^\mu_{\mathrm{CC}} &\equiv -\mathrm{i}\frac{g_W}{\sqrt{2}}\bar{u}(d)\gamma^\mu P_L u(u)\cdot V^*_{ud} \\
j^\mu_{\mathrm{CC}} &\equiv -\mathrm{i}\frac{g_W}{\sqrt{2}}\bar{v}(d)\gamma^\mu P_L v(u)\cdot V^*_{ud} \\
j^\mu_{\mathrm{CC}} &\equiv -\mathrm{i}\frac{g_W}{\sqrt{2}}\bar{u}(d)\gamma^\mu P_L v(u)\cdot V^*_{ud} \\
j^\mu_{\mathrm{CC}} &\equiv -\mathrm{i}\frac{g_W}{\sqrt{2}}\bar{v}(d)\gamma^\mu P_L u(u)\cdot V^*_{ud}
\end{aligned} \tag{10.6}$$

and we need a "rule of thumb" to decide whether $V_{UD}$ or $V^*_{UD}$ shows up (here we generalize to all 9 elements of $V_{\mathrm{CKM}}$). Whenever the $U$-type quark is in the adjoint spinor (left) and the $D$-type quark in the ordinary spinor (right), it is $V_{UD}$. Whenever the $D$-type quark is in the adjoint spinor (left) and the $U$-type quark in the ordinary spinor (right), it is $V^*_{UD}$.

The CKM matrix can be parametrized in a zillion ways. The "standard parametrization"

$$\begin{aligned}
V_{\mathrm{CKM}} &= \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{31} & & -s_{31}e^{-\mathrm{i}\delta} \\ & 1 & \\ s_{31}e^{\mathrm{i}\delta} & & c_{31} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix} \\
&= \begin{pmatrix} c_{12}c_{31} & s_{12}c_{31} & s_{31}e^{-\mathrm{i}\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{31}e^{\mathrm{i}\delta} & c_{12}c_{23} - s_{12}s_{23}s_{31}e^{\mathrm{i}\delta} & s_{23}c_{31} \\ s_{12}s_{23} - c_{12}c_{23}s_{31}e^{\mathrm{i}\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{31}e^{\mathrm{i}\delta} & c_{23}c_{31} \end{pmatrix}
\end{aligned} \tag{10.7}$$

proceeds in exactly the same manner as in Sec. 9, and it is highly practical. Despite the similarity in setup, the angles and the phase are different now. To minimize the risk of confusion, we call them $\phi_{12}, \phi_{23}, \phi_{31}$, and $\delta_{\mathrm{CKM}}$, respectively.

Physicswise the important issue is that $\delta_{\mathrm{CKM}}$ generates CP-violation (now through quark-line CC weak interactions). For $\delta_{\mathrm{CKM}} \neq 0$ the upper-right element and the lower-left $2\times 2$ block are complex, while all other entries are real. Note that the Cabibbo rotation matrix discussed above is the upper-left $2 \times 2$ block of $V_{\text{CKM}}$. Obviously, for this to be true the requirements $c_{12}c_{31} \simeq c_{12}c_{23} - s_{12}s_{23}s_{31}e^{\mathrm{i}\delta}$ and $s_{12}c_{31} \simeq s_{12}c_{23} + c_{12}s_{23}s_{31}e^{\mathrm{i}\delta}$ both need to be satisfied (which is the case, see below). Of profound phenomenological importance are the unitarity relations

$$\begin{aligned}
|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \quad &\leftarrow \quad |\mathrm{row}_1|^2 = 1 \\
|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1 \quad &\leftarrow \quad |\mathrm{row}_2|^2 = 1 \\
|V_{td}|^2 + |V_{ts}|^2 + |V_{tb}|^2 = 1 \quad &\leftarrow \quad |\mathrm{row}_3|^2 = 1 \\
V_{ud}^* V_{cd} + V_{us}^* V_{cs} + V_{ub}^* V_{cb} = 0 \quad &\leftarrow \quad \mathrm{row}_1 \perp \mathrm{row}_2 \\
\ldots \quad &\leftarrow \quad \mathrm{row}_2 \perp \mathrm{row}_3 \\
\ldots \quad &\leftarrow \quad \mathrm{row}_3 \perp \mathrm{row}_1 \\
\ldots \quad &\leftarrow \quad \mathrm{col}_1 \perp \mathrm{col}_2 \\
\ldots \quad &\leftarrow \quad \mathrm{col}_2 \perp \mathrm{col}_3 \\
V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0 \quad &\leftarrow \quad \mathrm{col}_3 \perp \mathrm{col}_1
\end{aligned} \tag{10.8}$$

where math tells us that the three conceivable normalization conditions for the columns provide no new information. In Fig. 14.25 of the book we see the use case of the last condition (which is the phenomenologically relevant one). Divide this relation by $V_{cb}^* V_{cd}$, so that it takes the form

$$\boxed{\frac{V_{ub}^* V_{ud}}{V_{cb}^* V_{cd}} + 1 + \frac{V_{tb}^* V_{td}}{V_{cb}^* V_{cd}} = 0} \tag{10.9}$$

and think of the three complex numbers as a vectors in the 2-dimensional $(\rho, \eta)$-plane. Starting with the apex (top corner), the relation states the closedness of the *unitarity triangle* of the SM. Note that the *area* of this triangle is proportional to the amount of CP-violation ($\delta_{\text{CKM}} \propto \eta$). The angle $\alpha$ (near the apex) is very close to a right angle, but in the SM it is a combination of fundamental parameters (with arbitrary values) which by chance yields $\alpha \simeq 90°$.

The magnitude of the elements is found (in experiment) to be

$$|V_{\text{CKM}}| = \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} \simeq \begin{pmatrix} 0.974 & 0.225 & 0.004 \\ 0.225 & 0.973 & 0.041 \\ 0.009 & 0.040 & 0.999 \end{pmatrix} \tag{10.10}$$

whereupon the CKM matrix is *diagonally dominated* (in contradistinction to the MNS matrix). As a result of this structure, it makes (here) sense to introduce the Wolfenstein parametrization

$$V_{\text{CKM}} = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - \mathrm{i}\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - \mathrm{i}\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4) \tag{10.11}$$

which, in principle, is an infinite series in powers of $\lambda$. Note that at $O(\lambda^3)$ only two elements ($V_{ub}$ and $V_{td}$) are complex, in contradistinction to the standard parametrization. Of course, going to higher order will eventually establish complete agreement.

## 10.4 Summary

- Unitary matrix $V_{\text{CKM}}$ parametrizes mixing among the $D$-type quarks ($\phi_{12}, \phi_{23}, \phi_{31}$ and $\delta_{\text{CKM}}$).
- $V_{\text{CKM}}$ is diagonally dominated ($2 \times 2$ block is close to Cabibbo rotation), while $U_{\text{MNS}}$ is not.
- Standard unitarity triangle captures $\mathrm{col}_3 \perp \mathrm{col}_1$, its *area* measures amount of CP violation.

# 11 Electroweak unification

## 11.1 $W$-decay and unitarity in $WW$ pair-production

If the $W$ would appear as an external particle, we would use the three polarizations

$$\epsilon^{(+)\mu} = -\frac{1}{\sqrt{2}}(0,1,\mathrm{i},0)\ , \quad \epsilon^{(0)\mu} = \frac{1}{m_W}(p,0,0,E)\ , \quad \epsilon^{(-)\mu} = +\frac{1}{\sqrt{2}}(0,1,-\mathrm{i},0) \tag{11.1}$$

(positive and negative helicity, transversal) to define the invariant matrix elements

$$-\mathrm{i}M_{fi}^{(\lambda)} = \epsilon_\mu^{(\lambda)}(p_1)\,\bar{u}(p_3)[-\mathrm{i}\frac{g_W}{\sqrt{2}}\gamma^\mu P_L]v(p_4) \tag{11.2}$$

for the kinematics shown in Fig. 15.1 of the book with $p_3^2\!=\!m_e^2$, $p_4^2\!\simeq\!0$ and $\lambda \in \{+,0,-\}$. With

$$M_{fi}^{(\lambda)} = \frac{g_W}{\sqrt{2}}\,\epsilon_\mu^{(\lambda)}(p_1)\,j^\mu(p_3,p_4) \qquad \text{and} \qquad j^\mu(p_3,p_4) \equiv \bar{u}(p_3)[\gamma^\mu \frac{1}{2}(1-\gamma^5)]v(p_4) \tag{11.3}$$

an equivalent formulation is given which involves the lepton current $j^\mu$. Given $m_Z \simeq 80.4\,\mathrm{GeV}$, taking the electron and neutrino as massless particles is a good approximation. This yields

$$p_1^\bullet = (m_W,0,0,0)\ , \quad p_3^\bullet = (E, E\sin(\vartheta), 0, E\cos(\vartheta))\ , \quad p_4^\bullet = (E, -\vec{p}_3) \tag{11.4}$$

with $E \equiv \frac{1}{2}m_W$ in the Lab system, and $\vartheta$ the angle between the $e^-$ and the $z$-axis (see Fig. 15.2). Chirality selection rules imply $j^\mu(p_3,p_4) \longrightarrow \bar{u}_\downarrow(p_3)\gamma^\mu v_\uparrow(p_4)$, and a standard evaluation yields $j^\mu \longrightarrow m_W(0,-\cos(\vartheta),-\mathrm{i},\sin(\vartheta))$. Specifically, for the $W$ at rest $\epsilon^{(0)\mu} \longrightarrow (0,0,0,1)$, so

$$\begin{aligned}
M_{fi}^{(+)} &= \frac{g_W m_W}{2}(0,+1,-\mathrm{i},0)\cdot(0,-\cos,-\mathrm{i},\sin) = \frac{1}{2}g_W m_W(1+\cos(\vartheta)) \\
M_{fi}^{(0)} &= \frac{g_W m_W}{\sqrt{2}}(0,0,0,1)\cdot(0,-\cos,-\mathrm{i},\sin) = -\frac{1}{\sqrt{2}}g_W m_W \sin(\vartheta) \\
M_{fi}^{(-)} &= \frac{g_W m_W}{2}(0,-1,-\mathrm{i},0)\cdot(0,-\cos,-\mathrm{i},\sin) = \frac{1}{2}g_W m_W(1-\cos(\vartheta))
\end{aligned} \tag{11.5}$$

and taking the absolute square yields

$$|M^{(+)}|^2 = g_W^2 m_W^2\,\frac{1}{4}(1+\cos)^2\ , \quad |M^{(0)}|^2 = g_W^2 m_W^2\,\frac{1}{2}\sin^2\ , \quad |M^{(-)}|^2 = g_W^2 m_W^2\,\frac{1}{4}(1-\cos)^2$$

as illustrated in Fig. 15.3. If the initial $W$ was unpolarized, we have gotten to average

$$\begin{aligned}
\langle|M_{fi}|^2\rangle &= \frac{1}{3}\Big[|M^{(+)}|^2 + |M^{(0)}|^2 + |M^{(-)}|^2\Big] \\
&= \frac{1}{3}g_W^2 m_W^2\Big[\frac{1}{4}(1+\cos)^2 + \frac{1}{2}\sin^2 + \frac{1}{4}(1-\cos)^2\Big] = \frac{1}{3}g_W^2 m_W^2
\end{aligned} \tag{11.6}$$

which looks plausible, as it is isotropic. Integrating over the $4\pi$ sphere yields the partial width

$$\Gamma(W^- \to e^-\bar{\nu}_e) = \frac{p^*}{32\pi^2 m_W^2}\int \langle|M_{fi}|^2\rangle\,\mathrm{d}\Omega = \frac{p^*}{32\pi^2 m_W^2}\,\frac{1}{3}g_W^2 m_W^2\,4\pi = \frac{p^*}{24\pi}\,g_W^2 \tag{11.7}$$

where $p^*$ is the absolute value of the three-momentum of the $e^-$ (the $\bar{\nu}$ is not seen), so

$$\Gamma(W^- \to e^-\bar{\nu}_e) = \frac{1}{48\pi}\, g_W^2 m_W \tag{11.8}$$

for this decay channel. Using $\Gamma(W^- \to e^-\bar{\nu}_e) = \Gamma(W^- \to \mu^-\bar{\nu}_\mu) = \Gamma(W^- \to \tau^-\bar{\nu}_\tau)$ yields

$$\Gamma(W^- \to \ell^-\bar{\nu}_\ell, \text{ summed}) = 3\Gamma(W^- \to e^-\bar{\nu}_e) = \frac{1}{16\pi}\, g_W^2 m_W \tag{11.9}$$

for the decay into any charged-uncharged lepton pair. But the $W^-$ may also decay into quarks

$$\begin{array}{lll} \Gamma(W^- \to d\bar{u}) = 3|V_{ud}|^2\Gamma_{e\bar{\nu}}\ , & \Gamma(W^- \to d\bar{c}) = 3|V_{cd}|^2\Gamma_{e\bar{\nu}}\ , & \Gamma(W^- \to d\bar{t}) = 0 \\ \Gamma(W^- \to s\bar{u}) = 3|V_{us}|^2\Gamma_{e\bar{\nu}}\ , & \Gamma(W^- \to s\bar{c}) = 3|V_{cs}|^2\Gamma_{e\bar{\nu}}\ , & \Gamma(W^- \to s\bar{t}) = 0 \\ \Gamma(W^- \to b\bar{u}) = 3|V_{ub}|^2\Gamma_{e\bar{\nu}}\ , & \Gamma(W^- \to b\bar{c}) = 3|V_{cb}|^2\Gamma_{e\bar{\nu}}\ , & \Gamma(W^- \to b\bar{t}) = 0 \end{array} \tag{11.10}$$

where we use that the coupling is still $g_W/\sqrt{2}$, see (11.2), for the flavor eigenstates, i.e. for $W^- \to d'\bar{u}$. We also take into account that there are 3 colors, and that the top is kinematically excluded ($m_W < m_t$). Using (10.8) the partial width for a $W$ decaying into any $D\bar{U}$-pair is

$$\Gamma(W^- \to D\bar{U}, \text{ summed}) = 6\Gamma(W^- \to e^-\bar{\nu}_e) = \frac{1}{8\pi}\, g_W^2 m_W \tag{11.11}$$

and including radiative corrections (see Fig. 14.4) amounts to equipping the hadronic contributions with a QCD enhancement factor $[1 + \alpha_S(m_W^2)/\pi] \simeq 1.038$. This yields the total width

$$\Gamma_W = \Big(3 + 6[1 + \alpha_S(m_W^2)/\pi]\Big)\Gamma(W^- \to e^-\bar{\nu}_e) \simeq 0.061\, g_W^2 m_W \simeq 2.1\,\mathrm{GeV} \tag{11.12}$$

and predicts the branching fraction for hadronic decays to be

$$B(W^- \to D\bar{U}) = \frac{6[1 + \alpha_S/\pi]}{3 + 6[1 + \alpha_S/\pi]} = 0.675\ . \tag{11.13}$$

By similar means the cross-section for $e^+e^- \to W^+W^-$ can be worked out. In Fig. 15.5 two contributions are shown we are familiar with ($\nu$-exchange in the $t$-channel, $\gamma$-exchange in the $s$-channel). The prediction with only $\nu$-exchange is *bigger* than with both $\nu$ and $\gamma$-exchange (dotted versus dashed line in Fig. 15.6). The SM knows a third contribution, $Z$-exchange in the $s$-channel (third panel in Fig. 15.5), and together they give the full line in Fig. 15.6. This is possible, since the 2nd and 3rd contribution interferes negatively with the 1st one

$$|M_\nu + M_\gamma + M_Z|^2 \ll |M_\nu + M_\gamma|^2 \ll |M_\nu|^2 \tag{11.14}$$

and indeed the data of LEP were found to lie on the lowest (full) line. Furthermore, there is also a theory argument which indicates that the theory with only the first (or the first two) contributions cannot be correct. These cross-sections *grow without bound* as a function of $\sqrt{s}$, thus violating unitarity at some $\sqrt{s^*}$ ("Froissart bound", see QFT course). It seems remarkable that an incomplete theory may thus point out its own limitations to the sapient physicist.

## 11.2 Weak interaction gauge group SU(2) in two representations

The Lie group $SU(2)_L$ of weak interactions acts on a field $\phi(x)$ as

$$\phi(x) \longrightarrow \phi'(x) \equiv \exp(\mathrm{i}g_W\vec{\alpha}(x)\vec{T})\phi(x) \qquad \text{with} \qquad \vec{T} \equiv \frac{1}{2}\vec{\sigma} \tag{11.15}$$

where the $T^a$ and thus exp(.) are $2 \times 2$ matrixes, hence it acts on a vector

$$\phi(x) \equiv \begin{pmatrix} \nu_e(x) \\ e^-(x) \end{pmatrix} \tag{11.16}$$

known as a "weak doublet" which transforms like an isospin doublet in strong interactions. Since the CC weak interaction affects only the L-chiralities of these particles, we should put only the $\nu_e(x)_L$ and $e^-(x)_L$ into this doublet. Since the $\nu_e(x)_R$ and $e^-(x)_R$ are unaffected, we must put them into a singlet. Hence, the leptonic part of the first generation is described as

$$\text{doublet} \quad \begin{pmatrix} \nu_e(x) \\ e^-(x) \end{pmatrix}_L , \qquad \text{singlet} \quad \nu_e(x)_R , \qquad \text{singlet} \quad e^-(x)_R . \tag{11.17}$$

In total we have 6 doublets under $SU(2)_L$ transforming in the *fundamental* representation

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L , \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L , \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L , \quad \begin{pmatrix} u \\ d' \end{pmatrix}_L , \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L , \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L \tag{11.18}$$

and 12 singlets under $SU(2)_L$ transforming in the *trivial* representation

$$\nu_{e,R} \ , \ \nu_{\mu,R} \ , \ \nu_{\tau,R} \ , \quad e^-_R \ , \ \mu^-_R \ , \ \tau^-_R \ , \quad u_R \ , \ c_R \ , \ t_R \ , \quad d_R \ , \ s_R \ , \ b_R \ . \tag{11.19}$$

In analogy with strong interactions, we also attribute them *weak isospin* $I = \frac{1}{2}$ in the doublet and $I = 0$ in the singlet, along with $I^3 = \pm\frac{1}{2}$ in the doublet and $I^3 = 0$ in the singlet. Note that the doublets contain *weak eigenstates* and thus account for the CKM mixings.

The requirement of local gauge invariance enforces the presence of the interaction term

$$\mathrm{i}g_W\gamma^\mu T^a \phi_L \, W^a_\mu \tag{11.20}$$

in the Dirac equation, where $a$ is an adjoint index of $SU(2)$ (so the implicit summation is $a = 1...3$, besides the one over $\mu$). Plugging everything in, we have the three weak currents

$$j_1^\mu \equiv g_W\bar{\phi}_L\gamma^\mu\frac{\sigma_1}{2}\phi_L \ , \qquad j_2^\mu \equiv g_W\bar{\phi}_L\gamma^\mu\frac{\sigma_2}{2}\phi_L \ , \qquad j_3^\mu \equiv g_W\bar{\phi}_L\gamma^\mu\frac{\sigma_3}{2}\phi_L \tag{11.21}$$

with $\phi_L \equiv (\nu_L \; e_L)^{\mathrm{trsp}}$ or $(u_L \; d_L)^{\mathrm{trsp}}$ a column-vector and $\bar{\phi}_L \equiv (\bar{\nu}_L \; \bar{e}_L)$ or $(\bar{u}_L \; \bar{d}_L)$ a row-vector in weak isospin space. To make the connection to physics clear, we focus on the term

$$\vec{j}^\mu\vec{W}_\mu = \sum_a j^\mu_a W^a_\mu \tag{11.22}$$

in the interaction part. Using $\sigma_\pm \equiv \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)$ it makes sense to introduce the currents

$$j^\mu_\pm \equiv \frac{g_W}{\sqrt{2}}\bar{\phi}_L\gamma^\mu\sigma_\pm\phi_L = \frac{g_W}{\sqrt{2}}\bar{\phi}_L\gamma^\mu\frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)\phi_L = \frac{1}{\sqrt{2}}\Big(j_1^\mu \pm \mathrm{i}j_2^\mu\Big) \tag{11.23}$$

which couple to the linear combinations

$$W^{\pm}_{\mu} \equiv \frac{1}{\sqrt{2}}\Big(W^1_{\mu} \mp \mathrm{i}W^2_{\mu}\Big) \tag{11.24}$$

of the gauge boson fields. This follows from

$$\begin{aligned} 2j^{\mu}_{+}W^{+}_{\mu} + 2j^{\mu}_{-}W^{-}_{\mu} &= (j^{\mu}_1 + \mathrm{i}j^{\mu}_2)(W^1_{\mu} - \mathrm{i}W^2_{\mu}) + (j^{\mu}_1 - \mathrm{i}j^{\mu}_2)(W^1_{\mu} + \mathrm{i}W^2_{\mu}) \\ &= j^{\mu}_1 W^1_{\mu} - \mathrm{i}j^{\mu}_1 W^2_{\mu} + \mathrm{i}j^{\mu}_2 W^1_{\mu} + j^{\mu}_2 W^2_{\mu} \\ &+ j^{\mu}_1 W^1_{\mu} + \mathrm{i}j^{\mu}_1 W^2_{\mu} - \mathrm{i}j^{\mu}_2 W^1_{\mu} + j^{\mu}_2 W^2_{\mu} \\ &= 2j^{\mu}_1 W^1_{\mu} + 2j^{\mu}_2 W^2_{\mu} \end{aligned} \tag{11.25}$$

and as a result we confirm the decomposition

$$\vec{j}^{\mu}\vec{W}_{\mu} \equiv \sum_a j^{\mu}_a W^a_{\mu} = j^{\mu}_{+}W^{+}_{\mu} + j^{\mu}_{-}W^{-}_{\mu} + j^{\mu}_3 W^3_{\mu} \tag{11.26}$$

whereupon the sum $a = 1, 2, 3$ has effectively been traded for a sum $a = +, -, 3$.

The current $j^{\mu}_{+}$ corresponds to an exchange of a $W^{+}$ boson and can be expressed as

$$\begin{aligned} j^{\mu}_{+} &= \frac{g_W}{\sqrt{2}}\bar{\phi}_L\gamma^{\mu}\sigma_{+}\phi_L = \frac{g_W}{\sqrt{2}}\begin{pmatrix}\bar{\nu}_L & \bar{e}_L\end{pmatrix}\gamma^{\mu}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} \\ &= \frac{g_W}{\sqrt{2}}\bar{\nu}_L\gamma^{\mu}e_L = \frac{g_W}{\sqrt{2}}\bar{\nu}\gamma^{\mu}P_L e = \frac{g_W}{\sqrt{2}}\bar{\nu}\gamma^{\mu}\frac{1}{2}(1-\gamma^5)e \end{aligned} \tag{11.27}$$

while $j^{\mu}_{-}$ corresponds to an exchange of a $W^{-}$ boson and can be expressed as

$$\begin{aligned} j^{\mu}_{-} &= \frac{g_W}{\sqrt{2}}\bar{\phi}_L\gamma^{\mu}\sigma_{-}\phi_L = \frac{g_W}{\sqrt{2}}\begin{pmatrix}\bar{\nu}_L & \bar{e}_L\end{pmatrix}\gamma^{\mu}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} \\ &= \frac{g_W}{\sqrt{2}}\bar{e}_L\gamma^{\mu}\nu_L = \frac{g_W}{\sqrt{2}}\bar{e}\gamma^{\mu}P_L\nu = \frac{g_W}{\sqrt{2}}\bar{e}\gamma^{\mu}\frac{1}{2}(1-\gamma^5)\nu \end{aligned} \tag{11.28}$$

and the respective vertices are displayed in Fig. 15.7. For the third current we obtain

$$\begin{aligned} j^{\mu}_3 &= \frac{g_W}{2}\bar{\phi}_L\gamma^{\mu}\sigma_3\phi_L = \frac{g_W}{2}\begin{pmatrix}\bar{\nu}_L & \bar{e}_L\end{pmatrix}\gamma^{\mu}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} \\ &= \frac{g_W}{2}\bar{\nu}_L\gamma^{\mu}\nu_L - \frac{g_W}{2}\bar{e}_L\gamma^{\mu}e_L = \frac{g_W}{2}\bar{\nu}\gamma^{\mu}\frac{1}{2}(1-\gamma^5)\nu - \frac{g_W}{2}\bar{e}\gamma^{\mu}\frac{1}{2}(1-\gamma^5)e \end{aligned} \tag{11.29}$$

with more terms than $j_{\pm}$. Using $I^3(\nu_e) = \frac{1}{2}$ and $I^3(e^-) = -\frac{1}{2}$ we may write this as

$$j^{\mu}_3 = I^3\, g_W\bar{\ell}\gamma^{\mu}\frac{1}{2}(1-\gamma^5)\ell \qquad \text{with} \qquad \ell \in \{\nu_e, e^-, \nu_{\mu}, \mu^-, \nu_{\tau}, \tau^-\} \tag{11.30}$$

hence packing the weak isospin assignments made above into the coupling allows for a significant simplification. The respective vertices are shown in Fig. 15.8. The $W^3$ is not the $Z^0$, but you may feel reminded of the lesson learned in GETA that the $Z^0$ does not change flavor.

## 11.3 Weak isospin and hypercharge for SU(2) doublets and singlets

The Lie Group $U(1)_Y$ of the hypercharge $Y$ (i.e. prior to $W^3$-$B$-mixing) acts on a field $\psi(x)$ as

$$\psi(x) \longrightarrow \psi'(x) \equiv U_Y(x)\psi(x) = \exp\Big(\mathrm{i}g_Y \frac{Y}{2}\beta(x)\Big)\psi(x) \tag{11.31}$$

resulting in an interaction term $\mathrm{i}g_Y \frac{y}{2}\gamma^\mu \psi B_\mu$ in the Dirac equation, where $y$ is the value of the operator $Y$ on the field $\psi$. This is similar to the $U(1)_Q \equiv U(1)_{\text{em}}$ transformation which gave raise to an interaction term $\mathrm{i}q\gamma^\mu\psi A_\mu$, except that $Q$ is replaced by $Y/2$.

As they share quantum numbers, there is nothing which prevents $W^3$ and $B$ from mixing

$$\boxed{\begin{array}{rcl} A_\mu &=& +B_\mu \cos(\theta_\mathrm{W}) + W^3_\mu \sin(\theta_\mathrm{W}) \\ Z_\mu &=& -B_\mu \sin(\theta_\mathrm{W}) + W^3_\mu \cos(\theta_\mathrm{W}) \end{array}} \tag{11.32}$$

which is accompanied by an analogous mixing of the currents ($\theta_\mathrm{W}$ is the “Weinberg angle”)

$$\boxed{\begin{array}{rcl} j^\mu_A &=& +j^\mu_Y \cos(\theta_\mathrm{W}) + j^\mu_3 \sin(\theta_\mathrm{W}) \\ j^\mu_Z &=& -j^\mu_Y \sin(\theta_\mathrm{W}) + j^\mu_3 \cos(\theta_\mathrm{W}) \end{array}}\,. \tag{11.33}$$

If we identify $A$ with the electromagnetic gauge potential, the current $j_A \equiv j_{\text{em}}$ must be the electromagnetic current. This gives the consistency requirement

$$j^\mu_{\text{em}} = \begin{cases} q_e \bar{e}_L \gamma^\mu e_L + q_e \bar{e}_R \gamma^\mu e_R \\ j^\mu_Y \cos(\theta_\mathrm{W}) + j^\mu_3 \sin(\theta_\mathrm{W}) \end{cases} \tag{11.34}$$

where $j^\mu_3$ was given in (11.29) and $j^\mu_Y$ follows from (11.31) to be

$$j^\mu_Y = \frac{g_Y}{2} Y_{e_L} \bar{e}_L \gamma^\mu e_L + \frac{g_Y}{2} Y_{e_R} \bar{e}_R \gamma^\mu e_R + \frac{g_Y}{2} Y_{\nu_L} \bar{\nu}_L \gamma^\mu \nu_L + \frac{g_Y}{2} Y_{\nu_R} \bar{\nu}_R \gamma^\mu \nu_R \tag{11.35}$$

where the book uses the abbreviation $g' \equiv g_Y$. Matching yields the four conditions

$$\begin{array}{lrl} \bar{e}_L\gamma^\mu e_L : & -e =& \frac{g_Y}{2} Y_{e_L} \cos(\theta_\mathrm{W}) - \frac{g_W}{2} \sin(\theta_\mathrm{W}) \\ \bar{e}_R\gamma^\mu e_R : & -e =& \frac{g_Y}{2} Y_{e_R} \cos(\theta_\mathrm{W}) \\ \bar{\nu}_L\gamma^\mu \nu_L : & 0 =& \frac{g_Y}{2} Y_{\nu_L} \cos(\theta_\mathrm{W}) + \frac{g_W}{2} \sin(\theta_\mathrm{W}) \\ \bar{\nu}_R\gamma^\mu \nu_R : & 0 =& \frac{g_Y}{2} Y_{\nu_R} \cos(\theta_\mathrm{W}) \end{array} \tag{11.36}$$

but this is not yet the full story. For invariance under $SU(2)_L$ *and* $U(1)_Y$ symmetry, the (weak) hypercharges of the states in a (weak) isospin-doublet *must be the same*: $Y_{e_L} = Y_{\nu_L}$. Due to the rotation, the weak hypercharge $Y$ must be a linear combination of electric charge $Q$ and weak 3-isospin $I^3$, say $Y = \alpha Q + \beta I^3$. The $e_L$ has $Q = -1, I^3 = -\frac{1}{2}$, and the $\nu_L$ has $Q = 0, I^3 = +\frac{1}{2}$. Hence the ansatz yields $Y_{e_L} = -\alpha - \frac{1}{2}\beta$ and $Y_{\nu_L} = \frac{1}{2}\beta$. The requirement that they be equal thus implies $-\alpha = \beta$, and with a purely conventional extra factor 2 we have

$$Y = 2\Big(Q - I^3\Big) \qquad \Longleftrightarrow \qquad \boxed{Q = I^3 + \frac{1}{2}Y} \tag{11.37}$$

which means that this formula looks exactly like the GellMann-Nishijima formula in strong interactions (see GETA and Sec. 6). Hence we can augment (11.17) to read

$$\boxed{\begin{pmatrix} \nu_e(x) \\ e^-(x) \end{pmatrix}_L \longleftrightarrow Y = -1 \quad , \quad \nu_e(x)_R \longleftrightarrow Y = 0 \quad , \quad e^-(x)_R \longleftrightarrow Y = -2} \tag{11.38}$$

together with $I^3 = \pm\frac{1}{2}$ and $I^3 = 0$ and $I^3 = 0$ in the three cases, respectively. Applying the GellMann-Nishijima formula gives $Q = \pm\frac{1}{2} - \frac{1}{2} \cdot 1 = 0, -1$ and $Q = 0 + \frac{1}{2} \cdot 0 = 0$ and $Q = 0 - \frac{1}{2} \cdot 2 = -1$ in the three cases, respectively, so everything is right.

With the hypercharge assignments $Y_L = -1$ for the left-handed doublet and $Y_R = (0, -2)$ for the singlets $(\nu_R, e_R)$ in hand, we go back to the first two lines of (11.36) and find

$$e = g_Y \cos(\theta_{\mathrm{W}}) \qquad \text{and} \qquad e = g_W \sin(\theta_{\mathrm{W}}) \ . \tag{11.39}$$

The numerical value of $\theta_{\mathrm{W}}$ has been determined in many experiments to be

$$\sin^2(\theta_{\mathrm{W}}) = 0.23146(12) \tag{11.40}$$

and together with (11.39) this implies

$$\boxed{\frac{\alpha_{\mathrm{em}}}{\alpha_W} = \frac{e^2}{g_W^2} = \sin^2(\theta_{\mathrm{W}}) \simeq 0.23} \ . \tag{11.41}$$

Hence we find that the weak coupling is actually *stronger* than the electromagnetic coupling.

## 11.4 Couplings of the $Z$ to quark and lepton currents

The current $j_Z$ that couples to the $Z$ follows from (11.33) and (11.29, 11.35) as

$$\begin{aligned} j_Z^\mu &= -g_Y \sin(\theta_{\mathrm{W}}) \Big[ (Q - I^3) \bar{u}_L \gamma^\mu u_L + Q \bar{u}_R \gamma^\mu u_R \Big] + g_W \cos(\theta_{\mathrm{W}}) I^3 [\bar{u}_L \gamma^\mu u_L] \\ &= \Big[ - g_Y \sin(\theta_{\mathrm{W}})(Q - I^3) + g_W \cos(\theta_{\mathrm{W}}) I^3 \Big] [\bar{u}_L \gamma^\mu u_L] - g_Y \sin(\theta_{\mathrm{W}}) Q [\bar{u}_R \gamma^\mu u_R] \end{aligned} \tag{11.42}$$

and with $g_Y = g_W \tan(\theta_{\mathrm{W}})$ it follows that

$$j_Z^\mu = g_W \Big[ - \frac{\sin^2(\theta_{\mathrm{W}})}{\cos(\theta_{\mathrm{W}})} (Q - I^3) + \cos(\theta_{\mathrm{W}}) I^3 \Big] [\bar{u}_L \gamma^\mu u_L] - g_W \frac{\sin^2(\theta_{\mathrm{W}})}{\cos(\theta_{\mathrm{W}})} Q [\bar{u}_R \gamma^\mu u_R] \ . \tag{11.43}$$

Hence by defining

$$g_Z \equiv \frac{g_W}{\cos(\theta_{\mathrm{W}})} = \frac{e}{\sin(\theta_{\mathrm{W}}) \cos(\theta_{\mathrm{W}})} = \frac{2e}{\sin(2\theta_{\mathrm{W}})} \tag{11.44}$$

the NC weak current (to which the $Z$ couples) can be written as

$$j_Z^\mu = g_Z \Big[ - \sin^2(\theta_{\mathrm{W}})(Q - I^3) + \cos^2(\theta_{\mathrm{W}}) I^3 \Big] [\bar{u}_L \gamma^\mu u_L] - g_Z \sin^2(\theta_{\mathrm{W}}) Q [\bar{u}_R \gamma^\mu u_R] \tag{11.45}$$

where the big square bracket is seen to be $-\sin^2(\theta_{\mathrm{W}}) Q + I^3$. Accordingly

$$\boxed{j_Z^\mu = g_Z \Big[ c_L \cdot \bar{u}_L \gamma^\mu u_L + c_R \cdot \bar{u}_R \gamma^\mu u_R \Big]} \qquad \text{with} \qquad \left. \begin{matrix} c_L \\ c_R \end{matrix} \right\} \equiv \left. \begin{matrix} I^3 \\ 0 \end{matrix} \right\} - \sin^2(\theta_{\mathrm{W}}) Q \tag{11.46}$$

and with the known value (11.40) one finds the first seven columns of the following table (recall that $Y_L$ is an assignment to the doublet, while the two $Y_R$ differ by two units from each other).

| fermion | $Q$ | $I^3_L$ | $I^3_R$ | $Y_L$ | $Y_R$ | $c_L$ | $c_R$ | $c_V$ | $c_A$ |
|---|---|---|---|---|---|---|---|---|---|
| $\nu_e, \nu_\mu, \nu_\tau$ | 0 | $+\frac{1}{2}$ | 0 | $-1$ | 0 | $+\frac{1}{2}$ | 0 | $+\frac{1}{2}$ | $+\frac{1}{2}$ |
| $e^-, \mu^-, \tau^-$ | $-1$ | $-\frac{1}{2}$ | 0 | $-1$ | $-2$ | $-0.27$ | $+0.23$ | $-0.04$ | $-\frac{1}{2}$ |
| $u, c, t$ | $+\frac{2}{3}$ | $+\frac{1}{2}$ | 0 | $+\frac{1}{3}$ | $+\frac{4}{3}$ | $+0.35$ | $-0.15$ | $+0.19$ | $+\frac{1}{2}$ |
| $d, s, b$ | $-\frac{1}{3}$ | $-\frac{1}{2}$ | 0 | $+\frac{1}{3}$ | $-\frac{2}{3}$ | $-0.42$ | $+0.08$ | $-0.35$ | $-\frac{1}{2}$ |

Using $\bar{u}_L\gamma^\mu u_L = \bar{u}\gamma^\mu\frac{1}{2}(1-\gamma^5)u$ and $\bar{u}_R\gamma^\mu u_R = \bar{u}\gamma^\mu\frac{1}{2}(1+\gamma^5)u$ the current $j^\mu_Z$ reads

$$j^\mu_Z = g_Z\bar{u}\gamma^\mu\Big[c_L\frac{1}{2}(1-\gamma^5) + c_R\frac{1}{2}(1+\gamma^5)\Big]u = g_Z\bar{u}\gamma^\mu\frac{1}{2}\Big[(c_L+c_R) - (c_L-c_R)\gamma^5\Big]u\;. \tag{11.47}$$

The most compact form is thus

$$\boxed{j^\mu_Z = \frac{g_Z}{2}\bar{u}\gamma^\mu\Big[c_V - c_A\gamma^5\Big]u} \qquad \text{with} \qquad \left.\begin{matrix} c_V \\ c_A \end{matrix}\right\} \equiv c_L \pm c_R \equiv I^3_{(L)} - \left\{\begin{matrix} 2\sin^2(\theta_{\rm W})Q \\ 0 \end{matrix}\right. \tag{11.48}$$

and this means that the Feynman rule for the $Z$-boson interaction vertex is

$$\boxed{-\mathrm{i}\frac{g_Z}{2}\gamma^\mu\Big[c_V - c_A\gamma^5\Big]} \tag{11.49}$$

which differs from the V-A structure of the $W$-interaction. Hence, the content of the last two columns of our table can be summarized as follows. Only the neutrinos couple to the $Z$ in a pure V-A fashion, whereas the charged leptons and quarks couple in a semi-complicated $[c_V \mp \frac{1}{2}\gamma^5]$ fashion, where "semi" refers to $c_A = \mp\frac{1}{2}$, and "complicated" refers to $c_R = -\sin^2\theta_{\rm W}\cdot Q$.

## 11.5 Application to $Z$-decay branching fractions

In the book there is an argument why even with general $c_V, c_A$ coefficients certain chiralities do not contribute to $Z$-decay matrix elements. The first (unnumbered) equation in "Decays of the Z" is afflicted with several typos; the corrected version reads

$$\begin{aligned} \bar{u}_R\gamma^\mu[c_V - c_A\gamma^5]v_R &= (P_Ru)^\dagger\gamma^0\gamma^\mu[c_V - c_A\gamma^5]P_Lv \;=\; u^\dagger P_R\gamma^0\gamma^\mu[c_V - c_A\gamma^5]P_Lv \\ &= u^\dagger\gamma^0 P_L\gamma^\mu P_L[c_V - c_A\gamma^5]v \;=\; \bar{u}\gamma^\mu P_RP_L[c_V - c_A\gamma^5]v \;=\; 0 \end{aligned} \tag{11.50}$$

and similarly for $\bar{u}_L\gamma^\mu[c_V - c_A\gamma^5]v_L$. Based on this it follows that

$$\Gamma(Z\to f\bar{f}) = \frac{g_Z^2 m_Z}{48\pi}(c_V^2 + c_A^2) \tag{11.51}$$

for a given fermion $f\in\{\nu_e, e, u, d\}$ or siblings in the 2nd or 3rd generation.

The value $m_Z$ is known and with (11.40) it follows that

$$g_Z^2 = \frac{g_W^2}{\cos^2(\theta_{\rm W})} = \frac{8m_W^2}{\sqrt{2}\cos^2(\theta_{\rm W})}G_F \simeq 0.55 \tag{11.52}$$

and this means that we can evaluate the decay width for $Z\to\nu_e\bar{\nu}_e$ with the table above as

$$\Gamma(Z\to\nu_e\bar{\nu}_e) = \frac{g_Z^2 m_Z}{48\pi}\Big(\frac{1}{4} + \frac{1}{4}\Big) \simeq 167\,\mathrm{MeV}\;. \tag{11.53}$$

Repeating this for all other $f$ in the SM, and summing over $f$ one gets the full width

$$\Gamma_Z \equiv \sum_f \Gamma(Z \to f\bar{f}) \tag{11.54}$$

and it is clear that one should include all fermions listed above, except for the top quark, since $m_Z < m_t$. Since all other fermions satisfy $m_f \ll m_Z$, treating them as massless is a good approximation. Using this generation symmetry (except for the top, of course) we find

$$\Gamma_Z = 3\Gamma(Z \to \nu\bar{\nu}) + 3\Gamma(Z \to e^- e^+) + 6\Gamma(Z \to u\bar{u}) + 9\Gamma(Z \to d\bar{d}) \tag{11.55}$$

where the third prefactor accounts for 2 generations and 3 colors, while the last one accounts for 3 generations and 3 colors. To make things even more correct, one may multiply the hadronic decay widths with the QCD enhancement factor $[1 + \alpha_S(m_W^2)/\pi] \simeq 1.038$. The essential part is the numbers for $c_V^2 + c_A^2$ that we may take from the table. This yields the ratios

$$3(0.25 + 0.25) \;\div\; 3(0.0016 + 0.25) \;\div\; 6(0.0361 + 0.25) \;\div\; 9(0.1225 + 0.25) \tag{11.56}$$

for the partial widths (with factors 1.03081 in the last two slots). This yields the predictions

$$\begin{aligned} B(Z \to \nu\bar{\nu}) &= 3 \cdot 6.9\% \qquad B(Z \to u\bar{u}) = B(Z \to c\bar{c}) = 12\% \\ B(Z \to \ell\bar{\ell}) &= 3 \cdot 3.5\% \qquad B(Z \to d\bar{d}) = B(Z \to s\bar{s}) = B(Z \to b\bar{b}) = 15\% \end{aligned} \tag{11.57}$$

for the branching fractions, and experiment finds them well confirmed. This is an important sanity check for the numbers in the table above. Note that the $3 \times 3$ block of "odd numbers" in the table stems from simple numbers (in the first four numerical columns) times a Weinberg factor $\sin^2(\theta_\mathrm{W})$, so things are not as complicated as they may look in the first place.

## 11.6 Summary

- Start with local gauge group $SU(2)_L \times U(1)_Y$ and couplings $g_W, g_W, g_W, g_Y$ per generator.
- Weinberg mixing $W^3, B \to Z, A$ brings modification of the last two as $g_W, g_Y \to g_Z, e$.
- Overall relationship is $e/\sin(\theta_\mathrm{W}) = g_W = g_Z \cos(\theta_\mathrm{W})$ which implies $e < g_W < g_Z$.
- Charged-current (CC) weak interactions had pure V-A structure: $-\mathrm{i}\frac{g_W}{\sqrt{2}}\gamma^\mu[\frac{1}{2} - \frac{1}{2}\gamma^5]$.
- Neutral-current (NC) weak interactions have complicated structure: $-\mathrm{i}\frac{g_Z}{2}\gamma^\mu[c_V - c_A\gamma^5]$.
- Coefficients $c_L$, $c_R$ follow from weak $I^3$, $Y$ via $c_L = I^3 - \sin^2(\theta_\mathrm{W})Q$ and $c_R = -\sin^2(\theta_\mathrm{W})Q$.
- Coefficients $c_V$, $c_A$ are defined through $c_V \equiv c_L + c_R$ and $c_A \equiv c_L - c_R$.
- R/L-chiralities have a joint $Q$, but weak $I^3_{L,R}$ differ by one half, and $Y_{L,R}$ differ by one unit.
- Weak GellMann-Nishijima formula mimics strong counterpart: $Q = I^3 + \frac{1}{2}Y$

# 12 Outlook

In principle there are two more broad topic which would be worthy of a detailed discussion

- A brief account of the Higgs mechanism
- An account on strategies to test possible loopholes of the SM

but time restrictions prevent us from doing so. I strongly recommend reading sections 16-18 of the book; they present these topics in a very nice and concise manner.